Computation of Nash Equilibria: Multi-Player Games

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Outline

- **Normal-form games**
  - Definitions
  - Algorithms and Complexity

- **Graphical games**
  - Definitions
  - Polymatrix games
    - Algorithms and Complexity
  - Graphical/polymatrix games: Recent tight results

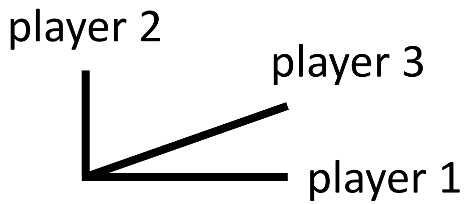
Normal-form games: Definitions

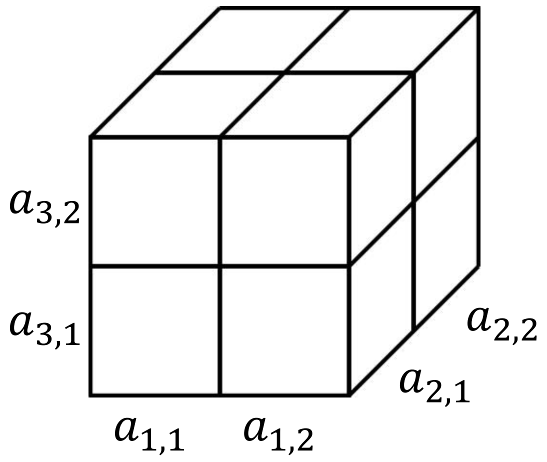
Normal-form games

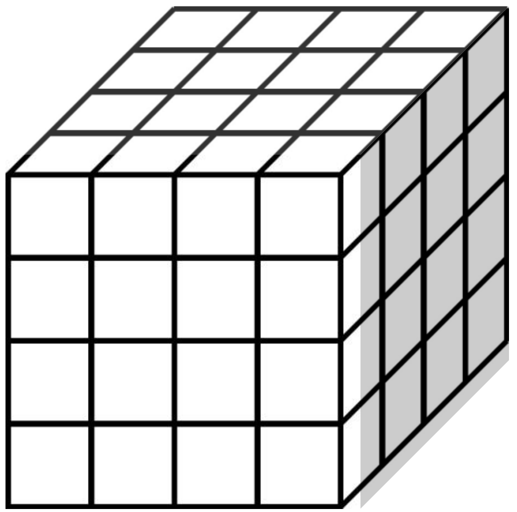
$2 \times 2 \times 2$

$m_1 \times m_2 \times m_3$

3 players:





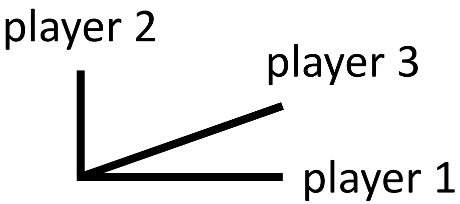


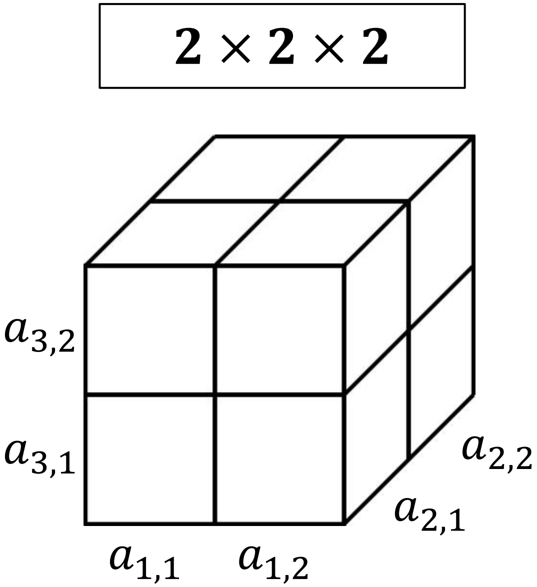
n players: Action profiles described by an $m_1 \times m_2 \times \cdots \times m_n$ tensor

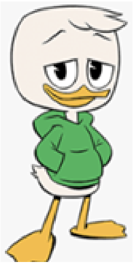
Input (n-player m-action game): $n \cdot m^n$ payoff entries

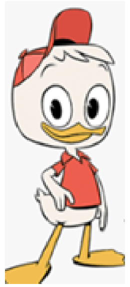
Normal-form games - visualization

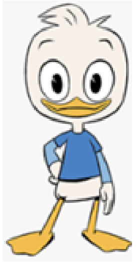
3 players:











U

	L	R
N	2, 4, 0	1, 4, 7
S	1, 2, 6	3, 2, -1

D

	L	R
N	1, 6, 0	1, 4, 0
S	2, 7, 4	3, 0, -1

Actions and strategies

n players, $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ **game**
They **simultaneously** choose actions:

- Player 1 chooses action $\boldsymbol{i_1} \in [\boldsymbol{m_1}]$

  ⋮

- Player j chooses action $\boldsymbol{i_j} \in [\boldsymbol{m_j}]$

  ⋮

- Player n chooses action $\boldsymbol{i_n} \in [\boldsymbol{m_n}]$

They can choose an action *probabilistically*!

- **Player $\boldsymbol{j}$** $\in [\boldsymbol{n}]$ chooses his action according to probability distribution $\boldsymbol{x_j}$
  - $\boldsymbol{x_{j,i_j}}$ is the probability he chooses action $\boldsymbol{i_j}$
    $\boldsymbol{x_{j,1} + x_{j,2} + \ldots + x_{j,m_j} = 1}$; $\boldsymbol{x_{j,i_j} \geq 0}$
  - $\boldsymbol{x_j}$ is the strategy of Player j
  - $(\boldsymbol{x_1, x_2 \ldots, x_n})$ is the strategy profile
  - $\boldsymbol{x_j}$ is an *action* (a.k.a. *pure strategy*) if $\boldsymbol{x_{j,i_j} = 1}$ for some $\boldsymbol{i_j} \in [\boldsymbol{m_j}]$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	1, 0, 2	3, −1, 2	4, 2, 0	−2, 0, 1
$x_{1,2}$ M	−1, 1, 0	1, 2, 4	−1, 1, −1	0, 1, 2
$x_{1,3}$ S	1, 2, −2	0, 1, 3	1, −2, 3	2, 2, 0

L $x_{3,2}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	3, 1, 1	2, 1, 1	0, −1, 2	−3, 1, −1
$x_{1,2}$ M	2, −1, 1	−2, 0, −2	2, 0, 1	0, −3, 2
$x_{1,3}$ S	−2, 0, 3	−1, 0, 2	−1, −1, −2	1, −1, −2

Actions and strategies

3 players, $\boldsymbol{m_1 \times m_2 \times m_3}$ **game**

They simultaneously choose actions:

- Player 1 chooses action $\boldsymbol{i_1} \in [\boldsymbol{m_1}]$ $\quad \boldsymbol{m_1 = 3}$

  ⋮
- Player j chooses action $\boldsymbol{i_j} \in [\boldsymbol{m_j}]$ $\quad \boldsymbol{m_2 = 4}$

  ⋮
- Player n chooses action $\boldsymbol{i_n} \in [\boldsymbol{m_n}]$ $\quad \boldsymbol{m_3 = 2}$

They can choose an action *probabilistically*!

- **Player** $\boldsymbol{j} \in [\boldsymbol{n}]$ chooses his action according to probability distribution $\boldsymbol{x_j}$
  - $\boldsymbol{x_{j,i_j}}$ is the probability he chooses action $\boldsymbol{i_j}$

    $\boldsymbol{x_{j,1} + x_{j,2} + \ldots + x_{j,m_j} = 1 \quad ; \quad x_{j,i_j} \geq 0}$
  - $\boldsymbol{x_j}$ is the strategy of Player j
  - $(\boldsymbol{x_1, x_2 \ldots, x_n})$ is the strategy profile
  - $\boldsymbol{x_j}$ is an *action* (a.k.a. *pure strategy*) if $\boldsymbol{x_{j,i_j} = 1}$ for some $\boldsymbol{i_j} \in [\boldsymbol{m_j}]$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$ M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2}$

Expected payoffs

n players, $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ **game**

- Let Player $j \in [n]$ **have payoff tensor P_j**
- In strategy profile $(x_1, x_2, \dots, x_n)$ the **expected payoff** of Player j is

$$\sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} x_{1,i_1} \cdot x_{2,i_2} \cdots x_{n,i_n} \cdot P_j(i_1, i_2, \dots, i_n)$$

$$= \langle P_j, x_1, x_2, \dots, x_n \rangle$$

For Player $j \in [n]$ and some $i_j \in [m_j]$:
$\langle P_j, x_{-j} \rangle_{i_j} \coloneqq \langle P_j, x_1, x_2, \dots, x_n \rangle_{x_{j,i_j}=1}$

$$= \sum_{i_j=1}^{m_j} x_{j,i_j} \langle P_j, x_{-j} \rangle_{i_j}$$

partial strategy profile

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$ M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2}$

Expected payoffs

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- Let Player $j \in [n]$ **have payoff tensor P_j**
- In strategy profile $(x_1, x_2, \ldots, x_n)$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} x_{j,i_j} \cdot \langle P_j, x_{-j} \rangle_{i_j}$$

Expected payoff of Player j when playing i_j

$\langle P_1, x_{-1} \rangle_1$

		$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$	N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$	M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$	S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1}$

		$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
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$x_{1,2}$	M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$	S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2}$

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Expected payoff of Player j when playing i_j

$\langle P_1, x_{-1} \rangle_1$

H $x_{3,1}$

		$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$1 = x_{1,1}$	N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$0 = x_{1,2}$	M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$0 = x_{1,3}$	S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

L $x_{3,2}$

		$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$1 = x_{1,1}$	N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$0 = x_{1,2}$	M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$0 = x_{1,3}$	S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

Expected payoffs

$\boldsymbol{n}$ **players,** $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ **game**

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- In strategy profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} x_{j,i_j} \cdot \langle P_j, x_{-j} \rangle_{i_j}$$

Expected payoff of Player j when playing i_j

$$\langle P_1, x_{-1} \rangle_1 =$$
$$1 \cdot x_{2,1} \cdot x_{3,1} + 3 \cdot x_{2,2} \cdot x_{3,1} + 4 \cdot x_{2,3} \cdot x_{3,1} - 2 \cdot x_{2,4} \cdot x_{3,1}$$
$$+3 \cdot x_{2,1} \cdot x_{3,2} + 2 \cdot x_{2,2} \cdot x_{3,2} + 0 \cdot x_{2,3} \cdot x_{3,2} - 3 \cdot x_{2,4} \cdot x_{3,2}$$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$1 = x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
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$0 = x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

L $x_{3,2}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$1 = x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$0 = x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$0 = x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

Expected payoffs

n **players,** $m_1 \times m_2 \times \cdots \times m_n$ **game**

- Let Player $j \in [n]$ **have payoff tensor** P_j
- In strategy profile $(x_1, x_2, \dots, x_n)$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} x_{j,i_j} \cdot \langle P_j, x_{-j} \rangle_{i_j}$$

Expected payoff of Player j when playing i_j

$\langle P_1, x_{-1} \rangle_1 =$

$$1 \cdot x_{2,1} \cdot x_{3,1} + 3 \cdot x_{2,2} \cdot x_{3,1} + 4 \cdot x_{2,3} \cdot x_{3,1} - 2 \cdot x_{2,4} \cdot x_{3,1}$$
$$+3 \cdot x_{2,1} \cdot x_{3,2} + 2 \cdot x_{2,2} \cdot x_{3,2} + 0 \cdot x_{2,3} \cdot x_{3,2} - 3 \cdot x_{2,4} \cdot x_{3,2}$$

$\vdots$

$\langle P_3, x_{-3} \rangle_2$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$ M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2}$

Expected payoffs

n players, $m_1 \times m_2 \times \cdots \times m_n$ **game**

- Let Player $j \in [n]$ **have payoff tensor P_j**
- In strategy profile $(x_1, x_2, \dots, x_n)$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} x_{j,i_j} \cdot \langle P_j, x_{-j} \rangle_{i_j}$$

Expected payoff of Player j when playing i_j

$$\langle P_1, x_{-1} \rangle_1 =$$
$$1 \cdot x_{2,1} \cdot x_{3,1} + 3 \cdot x_{2,2} \cdot x_{3,1} + 4 \cdot x_{2,3} \cdot x_{3,1} - 2 \cdot x_{2,4} \cdot x_{3,1}$$
$$+3 \cdot x_{2,1} \cdot x_{3,2} + 2 \cdot x_{2,2} \cdot x_{3,2} + 0 \cdot x_{2,3} \cdot x_{3,2} - 3 \cdot x_{2,4} \cdot x_{3,2}$$

$$\vdots$$

$$\langle P_3, x_{-3} \rangle_2$$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$ M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1} = 0$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2} = 1$

Expected payoffs

$\boldsymbol{n}$ **players,** $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ **game**

- Let Player $j \in [n]$ **have payoff tensor** $\boldsymbol{P_j}$
- In strategy profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} x_{j,i_j} \cdot \langle P_j, x_{-j} \rangle_{i_j}$$

Expected payoff of Player j when playing i_j

$$\langle P_1, x_{-1} \rangle_1 =$$
$$1 \cdot x_{2,1} \cdot x_{3,1} + 3 \cdot x_{2,2} \cdot x_{3,1} + 4 \cdot x_{2,3} \cdot x_{3,1} - 2 \cdot x_{2,4} \cdot x_{3,1}$$
$$+3 \cdot x_{2,1} \cdot x_{3,2} + 2 \cdot x_{2,2} \cdot x_{3,2} + 0 \cdot x_{2,3} \cdot x_{3,2} - 3 \cdot x_{2,4} \cdot x_{3,2}$$

$$\vdots$$

$$\langle P_3, x_{-3} \rangle_2 =$$
$$1 \cdot x_{1,1} \cdot x_{2,1} + 1 \cdot x_{1,1} \cdot x_{2,2} + 2 \cdot x_{1,1} \cdot x_{2,3} - 1 \cdot x_{1,1} \cdot x_{2,4}$$
$$+1 \cdot x_{1,2} \cdot x_{2,1} - 2 \cdot x_{1,2} \cdot x_{2,2} + 1 \cdot x_{1,2} \cdot x_{2,3} + 2 \cdot x_{1,2} \cdot x_{2,4}$$
$$+3 \cdot x_{1,3} \cdot x_{2,1} + 2 \cdot x_{1,3} \cdot x_{2,2} - 2 \cdot x_{1,3} \cdot x_{2,3} - 2 \cdot x_{1,3} \cdot x_{2,4}$$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	1, 0, 2	3, −1, 2	4, 2, 0	−2, 0, 1
$x_{1,2}$ M	−1, 1, 0	1, 2, 4	−1, 1, −1	0, 1, 2
$x_{1,3}$ S	1, 2, −2	0, 1, 3	1, −2, 3	2, 2, 0

H $x_{3,1} = 0$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	3, 1, 1	2, 1, 1	0, −1, 2	−3, 1, −1
$x_{1,2}$ M	2, −1, 1	−2, 0, −2	2, 0, 1	0, −3, 2
$x_{1,3}$ S	−2, 0, 3	−1, 0, 2	−1, −1, −2	1, −1, −2

L $x_{3,2} = 1$

Support, best responses, and regret

n **players,** $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ **game**

- Let Player $j \in [n]$ **have payoff tensor** $\boldsymbol{P_j}$
- In strategy profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$ the **expected payoff** of Player j is

$$\sum_{i_j=1}^{m_j} \boldsymbol{x_{j,i_j}} \cdot \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\boldsymbol{i_j}}$$

Expected payoff of Player j when playing i_j

Best responses

Given a partial strategy profile $\boldsymbol{x_{-j}}$ action $\boldsymbol{\hat{i}_j} \in [\boldsymbol{m_j}]$ is a *pure best response* if $\langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\hat{\boldsymbol{i}}_j} = \boldsymbol{max}_{\boldsymbol{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\boldsymbol{i_j}}$

Regret

The *regret* of Player j under a profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$ is

$$\boldsymbol{max}_{\boldsymbol{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\boldsymbol{i_j}} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n} \rangle$$

Support

For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x_j}$ is the subset of its actions played with positive probability

$$\boldsymbol{supp}(\boldsymbol{x_j}) = \{ \boldsymbol{i_j} \in [\boldsymbol{m_j}] : \boldsymbol{x_{j,i_j}} > \boldsymbol{0} \}$$

Support, best responses, and regret: example

Best responses
Given a partial strategy profile $\boldsymbol{x}_{-j}$ action $\hat{\boldsymbol{i}}_j \in [\boldsymbol{m}_j]$ is a *pure best response* if $\langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{\hat{i}_j} = \boldsymbol{max}_{i_j} \langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{i_j}$

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x}_j$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x}_j) = \{\boldsymbol{i}_j \in [\boldsymbol{m}_j] : \boldsymbol{x}_{j,i_j} > \boldsymbol{0}\}$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n)$ is

$$\boldsymbol{max}_{i_j} \langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{i_j} - \langle \boldsymbol{P}_j, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n \rangle$$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
$x_{1,2}$ M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
$x_{1,3}$ S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

H $x_{3,1}$

	$x_{2,1}$ A	$x_{2,2}$ B	$x_{2,3}$ C	$x_{2,4}$ D
$x_{1,1}$ N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
$x_{1,2}$ M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
$x_{1,3}$ S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

L $x_{3,2}$

Support, best responses, and regret: example

Best responses
Given a partial strategy profile $\boldsymbol{x_{-j}}$ action $\widehat{\boldsymbol{i}_j} \in [\boldsymbol{m_j}]$ is a *pure best response* if $\langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\widehat{i_j}} = \boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x_j}$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x_j}) = \{\boldsymbol{i_j} \in [\boldsymbol{m_j}] : \boldsymbol{x_{j,i_j}} > \boldsymbol{0}\}$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$ is

$$\boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n} \rangle$$

H $1/2$

		1/2 A	0 B	1/2 C	0 D
1	N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
0	M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
0	S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

L $1/2$

		1/2 A	0 B	1/2 C	0 D
1	N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
0	M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
0	S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

$\langle \boldsymbol{P_1}, \boldsymbol{x_{-1}} \rangle_N = \mathbf{2}$, $\quad \langle \boldsymbol{P_1}, \boldsymbol{x_{-1}} \rangle_M = \mathbf{1/2}$, $\quad \langle \boldsymbol{P_1}, \boldsymbol{x_{-1}} \rangle_S = \mathbf{-1/4}$

$\langle \boldsymbol{P_2}, \boldsymbol{x_{-2}} \rangle_A = \mathbf{1/2}$, $\quad \langle \boldsymbol{P_2}, \boldsymbol{x_{-2}} \rangle_B = \mathbf{0}$, $\quad \langle \boldsymbol{P_2}, \boldsymbol{x_{-2}} \rangle_C = \mathbf{1/2}$, $\quad \langle \boldsymbol{P_2}, \boldsymbol{x_{-2}} \rangle_D = \mathbf{1/2}$

$\langle \boldsymbol{P_3}, \boldsymbol{x_{-3}} \rangle_H = \mathbf{1}$, $\quad \langle \boldsymbol{P_3}, \boldsymbol{x_{-3}} \rangle_L = \mathbf{3/2}$

$$\langle \boldsymbol{P_1}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \rangle = \mathbf{2 \cdot 1 + 1/2 \cdot 0 - 1/2 \cdot 0 = 2}$$

Regret of Pl.1 $= \mathbf{2 - 2 = 0}$, **Regret of Pl.2** $= \mathbf{1/2 - 1/2 = 0}$, **Regret of Pl.3** $= \mathbf{3/2 - 5/4 = 1/4}$

Support, best responses, and regret: example

Best responses
Given a partial strategy profile $\boldsymbol{x_{-j}}$ action $\boldsymbol{\hat{i}_j} \in [\boldsymbol{m_j}]$ is a *pure best response* if $\langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\hat{i}_j} = \boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x_j}$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x_j}) = \{\boldsymbol{i_j} \in [\boldsymbol{m_j}] : \boldsymbol{x_{j,i_j}} > \boldsymbol{0}\}$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ is

$$\boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n} \rangle$$

H 1/2

		1/2 A	0 B	1/2 C	0 D
1	N	1, 0, 2	3, −1, 2	4, 2, 0	−2, 0, 1
0	M	−1, 1, 0	1, 2, 4	−1, 1, −1	0, 1, 2
0	S	1, 2, −2	0, 1, 3	1, −2, 3	2, 2, 0

L 1/2

		1/2 A	0 B	1/2 C	0 D
1	N	3, 1, 1	2, 1, 1	0, −1, 2	−3, 1, −1
0	M	2, −1, 1	−2, 0, −2	2, 0, 1	0, −3, 2
0	S	−2, 0, 3	−1, 0, 2	−1, −1, −2	1, −1, −2

$\langle P_1, x_{-1} \rangle_N = 2, \quad \langle P_1, x_{-1} \rangle_M = 1/2, \quad \langle P_1, x_{-1} \rangle_S = -1/4$

$\langle P_2, x_{-2} \rangle_A = 1/2, \quad \langle P_2, x_{-2} \rangle_B = 0, \quad \langle P_2, x_{-2} \rangle_C = 1/2, \quad \langle P_2, x_{-2} \rangle_D = 1/2$

$\langle P_3, x_{-3} \rangle_H = 1, \quad \langle P_3, x_{-3} \rangle_L = 3/2$

$$\langle P_2, x_1, x_2, x_3 \rangle = 1/2 \cdot 1/2 + 0 \cdot 0 + 1/2 \cdot 1/2 + 0 \cdot 1/2 = 1/2$$

Regret of Pl.1 $= 2 - 2 = 0$, **Regret of Pl.2** $= 1/2 - 1/2 = 0$, **Regret of Pl.3** $= 3/2 - 5/4 = 1/4$

Support, best responses, and regret: example

Best responses
Given a partial strategy profile $\boldsymbol{x}_{-j}$ action $\hat{\boldsymbol{i}}_j \in [\boldsymbol{m}_j]$ is a *pure best response* if $\langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{\hat{\boldsymbol{i}}_j} = \boldsymbol{max}_{\boldsymbol{i}_j} \langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{\boldsymbol{i}_j}$

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x}_j$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x}_j) = \{\boldsymbol{i}_j \in [\boldsymbol{m}_j] : \ \boldsymbol{x}_{j,\boldsymbol{i}_j} > \boldsymbol{0}\}$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n)$ is

$$\boldsymbol{max}_{\boldsymbol{i}_j} \langle \boldsymbol{P}_j, \boldsymbol{x}_{-j} \rangle_{\boldsymbol{i}_j} - \langle \boldsymbol{P}_j, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n \rangle$$

H $1/2$

		1/2 A	0 B	1/2 C	0 D
1	N	1, 0, 2	3, −1, 2	4, 2, 0	−2, 0, 1
0	M	−1, 1, 0	1, 2, 4	−1, 1, −1	0, 1, 2
0	S	1, 2, −2	0, 1, 3	1, −2, 3	2, 2, 0

L $1/2$

		1/2 A	0 B	1/2 C	0 D
1	N	3, 1, 1	2, 1, 1	0, −1, 2	−3, 1, −1
0	M	2, −1, 1	−2, 0, −2	2, 0, 1	0, −3, 2
0	S	−2, 0, 3	−1, 0, 2	−1, −1, −2	1, −1, −2

$\langle P_1, x_{-1} \rangle_N = 2, \quad \langle P_1, x_{-1} \rangle_M = 1/2, \quad \langle P_1, x_{-1} \rangle_S = -1/4$

$\langle P_2, x_{-2} \rangle_A = 1/2, \quad \langle P_2, x_{-2} \rangle_B = 0, \quad \langle P_2, x_{-2} \rangle_C = 1/2, \quad \langle P_2, x_{-2} \rangle_D = 1/2$

$\langle P_3, x_{-3} \rangle_H = 1, \quad \langle P_3, x_{-3} \rangle_L = 3/2$

$$\langle P_2, x_1, x_2, x_3 \rangle = 1/2 \cdot 1 + 1/2 \cdot 3/2 = 5/4$$

Regret of Pl.1 $= 2 - 2 = 0$, **Regret of Pl.2** $= 1/2 - 1/2 = 0$, **Regret of Pl.3** $= 3/2 - 5/4 = 1/4$

Nash equilibrium

Nash equilibrium
A strategy profile in which no player can gain more expected payoff by unilaterally changing her strategy

$\Updownarrow$ equivalent to any of:

Best responses
Given a partial strategy profile $\boldsymbol{x_{-j}}$ action $\boldsymbol{\hat{i}_j} \in [\boldsymbol{m_j}]$ is a *pure best response* if $\langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\hat{i}_j} = \boldsymbol{max_{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$

A strategy profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ in which every player is best-responding

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x_j}$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x_j}) = \{\boldsymbol{i_j} \in [\boldsymbol{m_j}] : \boldsymbol{x_{j,i_j}} > \boldsymbol{0}\}$

The **support** of every player $j \in [n]$ contains only **pure best responses**:

$$\boldsymbol{\hat{i}_j} \in \boldsymbol{supp}(\boldsymbol{x_j}) \implies \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\hat{i}_j} = \boldsymbol{max_{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ is

$$\boldsymbol{max_{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n} \rangle$$

The **regret** of every player $j \in [n]$ is $\boldsymbol{0}$:

$$\boldsymbol{max_{i_j}} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n} \rangle = \boldsymbol{0}$$

Nash equilibrium - existence

Nash equilibrium
A strategy profile in which no player can gain more expected payoff by unilaterally changing her strategy

Theorem [Nash, 1951]
Every finite game (finitely many players, finitely many actions per player) has at least one Nash equilibrium

Best responses
Given a partial strategy profile $\boldsymbol{x_{-j}}$ action $\widehat{\boldsymbol{\imath}}_{\boldsymbol{j}} \in [\boldsymbol{m_j}]$ is a *pure best response* if $\langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\widehat{\imath}_j} = \boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$

A strategy profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ in which every player is best-responding

Support
For some $j \in [n]$, the *support* of a strategy $\boldsymbol{x_j}$ is the subset of its actions played with positive probability $\boldsymbol{supp}(\boldsymbol{x_j}) = \{\boldsymbol{i_j} \in [\boldsymbol{m_j}] : \boldsymbol{x_{j,i_j}} > \boldsymbol{0}\}$

The **support** of every player $j \in [n]$ contains only **pure best responses**:

$$\widehat{\boldsymbol{\imath}}_{\boldsymbol{j}} \in \boldsymbol{supp}(\boldsymbol{x_j}) \implies \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{\widehat{\imath}_j} = \boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j}$$

Regret
The *regret* of Player j under a profile $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ is

$$\boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n} \rangle$$

The **regret** of every player $j \in [n]$ is $\boldsymbol{0}$:

$$\boldsymbol{max}_{i_j} \langle \boldsymbol{P_j}, \boldsymbol{x_{-j}} \rangle_{i_j} - \langle \boldsymbol{P_j}, \boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n} \rangle = \boldsymbol{0}$$

Nash equilibrium: example

A strategy profile $(x_1, x_2, \dots, x_n)$ in which every player is best-responding

The **support** of every player $j \in [n]$ contains only **pure best responses**:

$$\hat{i_j} \in supp(x_j) \implies \langle P_j, x_{-j} \rangle_{\hat{i_j}} = max_{i_j} \langle P_j, x_{-j} \rangle_{i_j}$$

The **regret** of every player $j \in [n]$ is **0**:

$$max_{i_j} \langle P_j, x_{-j} \rangle_{i_j} - \langle P_j, x_1, x_2, \dots, x_n \rangle = 0$$

H 1/2

		1/2 A	0 B	1/2 C	0 D
1	N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
0	M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
0	S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

L 1/2

		1/2 A	0 B	1/2 C	0 D
1	N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
0	M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
0	S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

$\langle P_1, x_{-1} \rangle_N = 2, \quad \langle P_1, x_{-1} \rangle_M = 1/2, \quad \langle P_1, x_{-1} \rangle_S = -1/4$

$\langle P_2, x_{-2} \rangle_A = 1/2, \quad \langle P_2, x_{-2} \rangle_B = 0, \quad \langle P_2, x_{-2} \rangle_C = 1/2, \quad \langle P_2, x_{-2} \rangle_D = 1/2$

$\langle P_3, x_{-3} \rangle_H = 1, \quad \langle P_3, x_{-3} \rangle_L = 3/2$

$supp(x_1) = \{N\}, \quad supp(x_2) = \{A, C\}, \quad supp(x_3) = \{H, L\}$

Regret of Pl.1 $= 2 - 2 = 0$, Regret of Pl.2 $= 1/2 - 1/2 = 0$, Regret of Pl.3 $= 3/2 - 5/4 = 1/4$

Nash equilibrium: example

A strategy profile $(x_1, x_2, \dots, x_n)$ in which every player is best-responding

The **support** of every player $j \in [n]$ contains only **pure best responses**:

$$\hat{i_j} \in supp(x_j) \implies \langle P_j, x_{-j} \rangle_{\hat{i_j}} = max_{i_j} \langle P_j, x_{-j} \rangle_{i_j}$$

The **regret** of every player $j \in [n]$ is **0**:

$$max_{i_j} \langle P_j, x_{-j} \rangle_{i_j} - \langle P_j, x_1, x_2, \dots, x_n \rangle = 0$$

It is an equilibrium!
(might be more)

H 1/2

		2/3 A	0 B	1/3 C	0 D
1	N	$1, 0, 2$	$3, -1, 2$	$4, 2, 0$	$-2, 0, 1$
0	M	$-1, 1, 0$	$1, 2, 4$	$-1, 1, -1$	$0, 1, 2$
0	S	$1, 2, -2$	$0, 1, 3$	$1, -2, 3$	$2, 2, 0$

L 1/2

		2/3 A	0 B	1/3 C	0 D
1	N	$3, 1, 1$	$2, 1, 1$	$0, -1, 2$	$-3, 1, -1$
0	M	$2, -1, 1$	$-2, 0, -2$	$2, 0, 1$	$0, -3, 2$
0	S	$-2, 0, 3$	$-1, 0, 2$	$-1, -1, -2$	$1, -1, -2$

$\langle P_1, x_{-1} \rangle_N = 2,$ $\langle P_1, x_{-1} \rangle_M = 1/2,$ $\langle P_1, x_{-1} \rangle_S = -1/3$

$\langle P_2, x_{-2} \rangle_A = 1/2,$ $\langle P_2, x_{-2} \rangle_B = 0,$ $\langle P_2, x_{-2} \rangle_C = 1/2,$ $\langle P_2, x_{-2} \rangle_D = 1/2$

$\langle P_3, x_{-3} \rangle_H = 4/3,$ $\langle P_3, x_{-3} \rangle_L = 4/3$

$supp(x_1) = \{N\},$ $supp(x_2) = \{A, C\},$ $supp(x_3) = \{H, L\}$

Regret of Pl.1 $= 2 - 2 = 0,$ Regret of Pl.2 $= 1/2 - 1/2 = 0,$ Regret of Pl.3 $= 4/3 - 4/3 = 0$

Notions of approximate Nash equilibria

Nash equilibrium
A strategy profile in which no player can gain more expected payoff by unilaterally changing her strategy

$\Updownarrow$

The **support** of every player $j \in [n]$ contains only **pure best responses**:
$$\hat{i}_j \in supp(x_j) \implies \langle P_j, x_{-j} \rangle_{\hat{i}_j} = max_{i_j} \langle P_j, x_{-j} \rangle_{i_j}$$

$\Longleftrightarrow$

The **regret** of every player $j \in [n]$ is $\mathbf{0}$:
$$max_{i_j} \langle P_j, x_{-j} \rangle_{i_j} - \langle P_j, x_1, x_2, \ldots, x_n \rangle = 0$$

ε-Well-Supported Nash equilibrium (ε-WSNE)

The **support** of every player $j \in [n]$ contains only **ε-best responses**:
$$\hat{i}_j \in supp(x_j) \implies \langle P_j, x_{-j} \rangle_{\hat{i}_j} \geq max_{i_j} \langle P_j, x_{-j} \rangle_{i_j} - \varepsilon$$

$\Longrightarrow$

ε-Nash equilibrium (ε-NE)

The **regret** of every player $j \in [n]$ is at most $\boldsymbol{\varepsilon}$:
$$max_{i_j} \langle P_j, x_{-j} \rangle_{i_j} - \langle P_j, x_1, x_2, \ldots, x_n \rangle \leq \varepsilon$$

0-WSNE = 0-NE = (exact) Nash equilibrium

- **Normalization: w.l.o.g. all payoffs in $[0, 1]$ $\longrightarrow$ $\varepsilon \in [0, 1]$**

Normal-form games: special families ($n \geq 2$)

- **Zero-sum (constant-sum):** the payoffs in each action profile sum to a fixed number

0 / 0	1 / -1
-1 / 1	0 / 0
1 / -1	-1 / 1

- **Symmetric:** all players have the same action set, and for every action profile $a = (a_1, \ldots, a_n)$ and every perturbation $\pi: [n] \to [n]$ we have

$$U_j(a_1, \ldots, a_n) = U_{\pi^{-1}(j)}\left(a_{\pi(1)}, \ldots, a_{\pi(n)}\right)$$

0 / 0	1 / -1	-1 / 1
-1 / 1	0 / 0	1 / -1
1 / -1	-1 / 1	0 / 0

❖ Find a **symmetric** Nash equilibrium $(\boldsymbol{s}, \boldsymbol{s}, \ldots, \boldsymbol{s})$ (there always exists one in symmetric games [a])

[a] Non-cooperative games. Nash. 1951

Normal-form games: special families ($n \geq 2$)

- **Win-lose:** payoffs in $\{0,1\}$

0, 0	1, 0
0, 1	0, 0
1, 0	1, 1

- **Coordination:** identical payoff tensors for all players

-2, -2	1, 1
0, 0	0, 0
1, 1	3, 3

Normal-form games:
Algorithms and Complexity

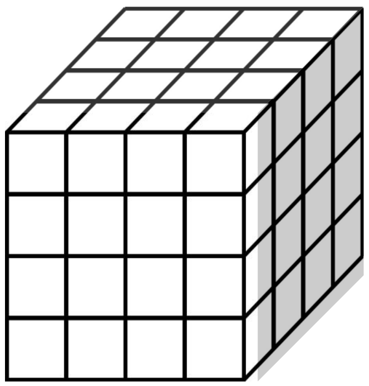
Computing *pure* vs computing *mixed* Nash equilibria

Pure Nash equilibrium (PNE)
An action profile in which no player can gain more payoff by unilaterally changing her action

- **Problem:** Given a normal-form game, find a PNE (if it exists) or decide non-existence
  - **What is the complexity?**
    - ✓ **Poly-time in the input size!**
      - For every action profile $(a_1, a_2, \dots, a_n)$:
      - Pick $j \in [n]$ and check if any of her $m - 1$ alternatives gives higher payoff: $n(m-1)$ comparisons for each of the m^n profiles

n players: Action profiles described by an $\boldsymbol{m_1 \times m_2 \times \cdots \times m_n}$ tensor

Input (n-player m-action game): $\boldsymbol{n \cdot m^n}$ payoff entries



We will focus on computing (approximate) mixed Nash equilibria

Hardness results for $n \geq 3$

- **Zero-sum (constant-sum):** the payoffs in each action profile sum to a fixed number

0, 0	1, -1
-1, 1	0, 0
1, -1	-1, 1

- **PPAD-complete**
  - ✓ By reduction from 2-player games: add a "dummy player" that makes payoffs sum to zero

- Recall from Argy's talk: 2-player zero-sum games are poly-time solvable!

Hardness results for $n \geq 3$

- **Symmetric:** all players have the same action set, and for every action profile $a = (a_1, \dots, a_n)$ and every perturbation $\pi: [n] \to [n]$ we have
$$U_j(a_1, \dots, a_n) = U_{\pi^{-1}(j)}\left(a_{\pi(1)}, \dots, a_{\pi(n)}\right)$$

0 / 0	1 / -1	-1 / 1
-1 / 1	0 / 0	1 / -1
1 / -1	-1 / 1	0 / 0

❖ Find a **symmetric** Nash equilibrium $(\boldsymbol{s}, \boldsymbol{s}, \dots, \boldsymbol{s})$ (there always exists one in symmetric games [a])

➢ **PPAD-complete**
  - ✓ Reduce from previous 3-player (zero-sum) game to a 3-player symmetric game [b]

❖ **Open problem**: What is the complexity of finding *any* (also non-symmetric) Nash equilibrium in a symmetric game?

[a] Non-cooperative games. Nash. 1951

[b] ∃R-Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria. Garg, Mehta, Vazirani, Yazdanbod. 2018

Hardness results for $n \geq 3$

- **Win-lose:** payoffs in $\{0,1\}$

0 0	1 0
0 1	0 0
1 0	1 1

➢ **PPAD-complete**

✓ Even 2-player win-lose games are PPAD-complete, for $\varepsilon = 1/poly(m)$ [a] (recall Argy's talk)

Do we have any "good" algorithms? For what ε?

[a] The approximation complexity of win-lose games. Chen, Teng, Valiant. 2007

Algorithms for $n \geq 3$

recall from Argy's talk

Algorithms for NE – support enumeration

(x, y) is an NE iff

$\hat{\imath} \in supp(x) \Rightarrow (Ry)_{\hat{\imath}} = max_i (Ry)_i$

$\hat{\jmath} \in supp(y) \Rightarrow (C^T x)_{\hat{\jmath}} = max_j (C^T x)_j$

$(Ry)_i = (Ry)_{\hat{\imath}}$ for every $i, \hat{\imath}$ in S
$(Ry)_i \geq (Ry)_{\hat{\imath}}$ for every $i \in S$ and $\hat{\imath} \notin S$
$\sum_i x_i = 1$
$x_i > 0$ for every $i \in S$
$x_i = 0$ for every $i \notin S$

$(C^T x)_j = (C^T x)_{\hat{\jmath}}$ for every $i, \hat{\imath}$ in T
$(C^T x)_j \geq (C^T x)_{\hat{\jmath}}$ for every $j \in T$ and $\hat{\jmath} \notin T$
$\sum_j y_j = 1$
$y_j > 0$ for every $j \in T$
$y_j = 0$ for every $j \notin T$

For each possible support S of Row player and each possible support T of Col player check if the linear system above has a feasible solution

$2^{m-1} \cdot 2^{n-1}$ choices!!

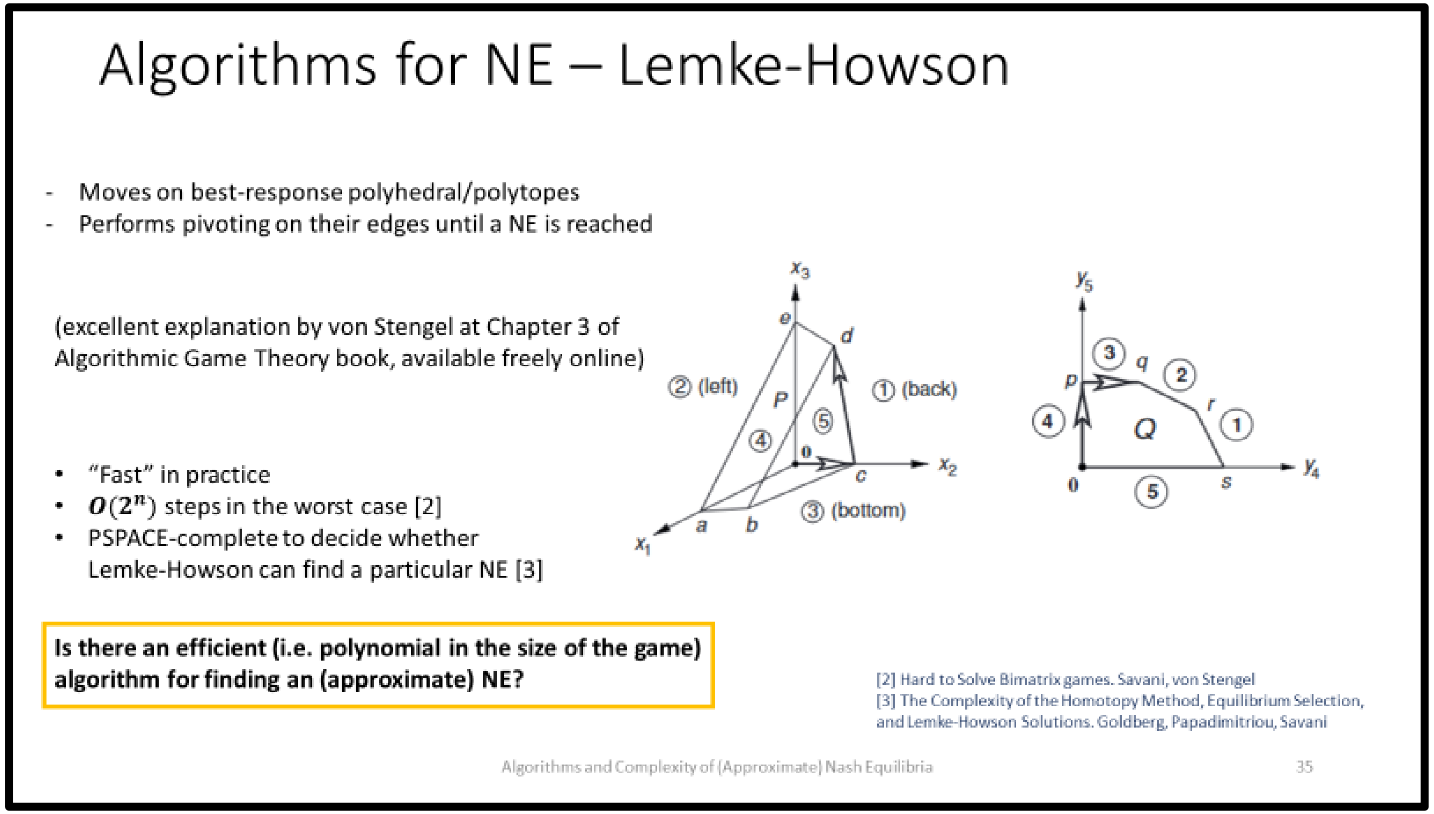
Every combination of sets corresponds to a feasibility problem that involves **linear** equations/inequalities

In principle, it can be used for **n-player** games

Every combination of sets corresponds to a feasibility problem that involves multilinear polynomial equations/inequalities of **degree $n - 1$**

Algorithms for $n \geq 3$

recall from Argy's talk



Algorithms for NE – Lemke-Howson

- Moves on best-response polyhedral/polytopes
- Performs pivoting on their edges until a NE is reached

(excellent explanation by von Stengel at Chapter 3 of Algorithmic Game Theory book, available freely online)

- "Fast" in practice
- $O(2^n)$ steps in the worst case [2]
- PSPACE-complete to decide whether Lemke-Howson can find a particular NE [3]

Is there an efficient (i.e. polynomial in the size of the game) algorithm for finding an (approximate) NE?

[2] Hard to Solve Bimatrix games. Savani, von Stengel
[3] The Complexity of the Homotopy Method, Equilibrium Selection, and Lemke-Howson Solutions. Goldberg, Papadimitriou, Savani

Extended for n-player games
[a], [b]

[a] Computing Equilibria of N-Person Games. Wilson. 1971

[b] On a Generalization of the Lemke-Howson Algorithm to Noncooperative N-Person Games. Rosenmuller. 1971

Algorithms for $n \geq 3$: Quasi-PTAS

recall from Argy's talk

A QPTAS for ϵ-NE

We can find an ϵ-NE in $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ time [29]

There always exists an ϵ-NE with support size $\log n/\epsilon^2$

- Take any pair of strategies (x, y)
- Randomly sample $\log n/\epsilon^2$ pure strategies
- Play the sampled strategies uniformly
- The resulting payoffs will be within ϵ of the originals w.h.p.

[29] Playing large games using simple strategies. Lipton, Markakis, Mehta

Algorithms and Complexity of (Approximate) Nash Equilibria 36

So, for constant n and ε, this algorithm is asymptotically tight!

Works also for n-player games
[a], [b]

It gives a QPTAS:

$$m^{O\left(n \cdot \frac{\log n + \log m - \log \varepsilon}{\varepsilon^2}\right)}$$

By [c], even for 2-player games, there is a constant $\varepsilon > 0$ such that any algorithm requires time

$$m^{O\left(\log^{1-o(1)} m\right)}$$

unless ETH for PPAD is false

Can we do better than quasi-poly-time for large enough ε?

[a] Playing large games using simple strategies. Lipton, Markakis, Mehta. 2003

[b] Empirical Distribution of Equilibrium and its Testing Application. Babichenko, Barman, Peretz. 2013

[c] Settling the complexity of computing approximate two-player Nash equilibria. Rubinstein. 2016

Algorithms for $n \geq 3$: Poly-time

recall from Argy's talk

For 2-player games, there is a poly-time algorithm that finds a $\left(\frac{1}{3} + \boldsymbol{\delta}\right)$**-NE** for any $\delta > 0$. [a]

For 2-player games, there is a poly-time algorithm that finds a $\left(\frac{1}{2} + \boldsymbol{\delta}\right)$**-WSNE** for any $\delta > 0$. [c]

Extension to n-player games:
By [b], if we have a poly-time algorithm that finds an ε_k-NE in any k-player game, then we can compute in poly-time an ε_{k+1}-NE for any $(k+1)$-player game, where $\varepsilon_{k+1} = \frac{1}{2-\varepsilon_k}$

3-player games: $\left(\frac{3}{5} + \delta\right)$-NE
4-player games: $\left(\frac{5}{7} + \delta\right)$-NE
⋮

❖ **Open problem**: For **3-player games**, is there is a poly-time algorithm that finds a ε-WSNE for some "small" $\varepsilon > 0$?

[a] A Polynomial-Time Algorithm for 1/3-Approximate Nash Equilibria in Bimatrix Games. Deligkas, Fasoulakis, Markakis. 2022

[b] New algorithms for approximate Nash equilibria in bimatrix games. Bosse, Byrka, Markakis. 2010

[c] A Polynomial-Time Algorithm for 1/2-Well-Supported Nash Equilibria in Bimatrix Games. Deligkas, Fasoulakis, Markakis. 2022

Hardness of computing constrained Nash equilibria

recall from Argy's talk

Complexity of constrained Nash equilibria

<u>Complexity Crash Course</u>

NP-complete
- YES/NO problems
- Verify in polynomial time any solution of the given problem

Polynomial-time algorithm is unlikely for NP-complete problems

$$max_i (Ry)_i - x^T Ry = 0$$
$$max_j (C^T x)_j - x^T Cy = 0$$

Problem definition
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\min(\mathbf{x}^T R\mathbf{y}, \mathbf{x}^T C\mathbf{y}) \geq u$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ with $\text{supp}(\mathbf{x}) \subseteq S$?
Are there two ϵ-NE with TV distance $\geq d$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ with $\max_i \mathbf{x}_i \leq p$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}^T R\mathbf{y} + \mathbf{x}^T C\mathbf{y} \leq v$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}^T R\mathbf{y} \leq u$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\lvert\text{supp}(\mathbf{x})\rvert + \lvert\text{supp}(\mathbf{y})\rvert \geq 2k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\min\{\lvert\text{supp}(\mathbf{x})\rvert, \lvert\text{supp}(\mathbf{y})\rvert\} \geq k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\lvert\text{supp}(\mathbf{x})\rvert \geq k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ with $S_R \subseteq \text{supp}(\mathbf{x})$?

It is NP-hard to decide whether a bimatrix game possesses an exact NE that satisfies any of the constraints above even for symmetric win-lose games [4], [5], [6]

[4] Nash and correlated equilibria: Some complexity considerations. Gilboa, Zemel
[5] New complexity results about Nash equilibria. Conitzer, Sandholm
[6] The complexity of Computational Problems about Nash Equilibria in Symmetric Win-Lose Games. Bilo, Mavronicolas

Implies **NP-completeness for n-player** games ($n \geq 3$): add "dummy" players to the 2-player game

Is there an "efficient" algorithm for constrained Nash equilibria?

An algorithm for computing constrained Nash equilibria

A **QPTAS** for computing constrained Nash equilibria:

Given an $\boldsymbol{n}$**-player game** with
- at most $\boldsymbol{m}$ **actions** per player
- $\boldsymbol{k}$ **many constraints** written as equalities/inequalities of polynomials with **maximum degree** $\boldsymbol{d}$

there is an algorithm that **either finds an** $\boldsymbol{\varepsilon}$**-constrained** $\boldsymbol{\varepsilon}$**-NE** in time

$$\boldsymbol{m}^{O\left(\frac{n^6 \cdot d^6 \cdot \log(2ndk)}{\varepsilon^5}\right)}$$

NE constraints:
$k = poly(n \cdot m)$

or answers that **there is no** $\mathbf{0}$**-constrained** $\mathbf{0}$**-NE** [a]

So, for constant n, d and ε, this algorithm is asymptotically tight!

The constrained problems at hand can be written as polynomial (in)equalities

By [b], even for 2-player games, for any $\varepsilon < 1/8$, any algorithm requires time

$$m^{O(\log m)}$$

unless ETH for 3SAT is false

[a] Approximating the existential theory of the reals. Deligkas, Fearnley, Melissourgos, Spirakis. 2018

[b] Inapproximability results for constrained approximate Nash equilibria. Deligkas, Fearnley, Savani. 2018

An intermission for *exact* Nash equilibria

Even for **3-player games**:

- finding an **exact NE** (i.e. 0-NE) is **FIXP-complete** [a]
- deciding existence of a **constrained exact NE** is **ETR-complete** [b], [c], [d], [e] even for symmetric games or zero-sum games

- **FIXP** contains the Sum-Of-Squares problem: not even known to be in NP [a]
- NP $\subseteq$ **ETR** $\subseteq$ PSPACE [f]

a.k.a. $\exists\mathbb{R}$

Still, as soon as we relax to ε-NE, the previous QPTAS applies!

[a] On the Complexity of Nash Equilibria and Other Fixed Points. Etessami, Yannakakis. 2010

[b] Fixed points, Nash equilibria, and the existential theory of the reals. Schaefer, Štefankovic. 2017

[c] ∃R-Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria. Garg, Mehta, Vazirani, Yazdanbod. 2018

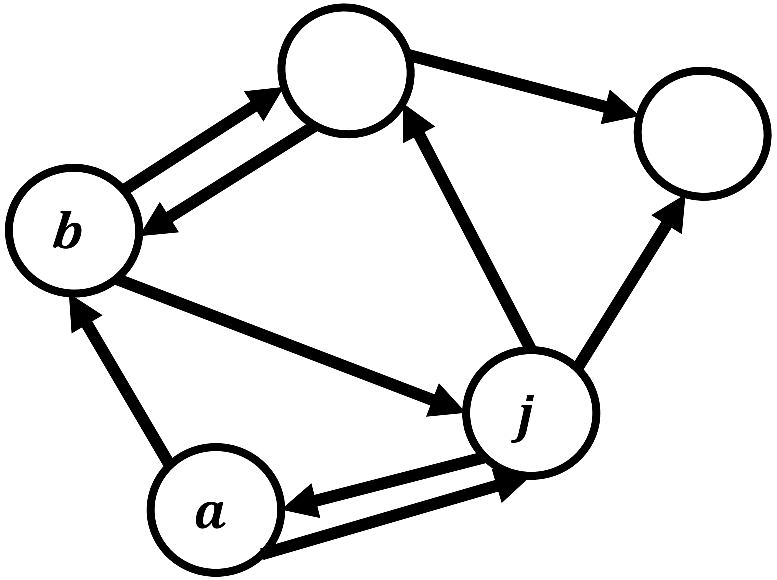
[d] ETR-complete decision problems about (symmetric) Nash equilibria in (symmetric) multi-player games. Bilò, Mavronicolas. 2017

[e] On the Computational Complexity of Decision Problems About Multi-player Nash Equilibria. Berthelsen, Hansen. 2022

[f] Some algebraic and geometric computations in PSPACE. Canny. 1988

Graphical games: Definitions

Graphical games [a]



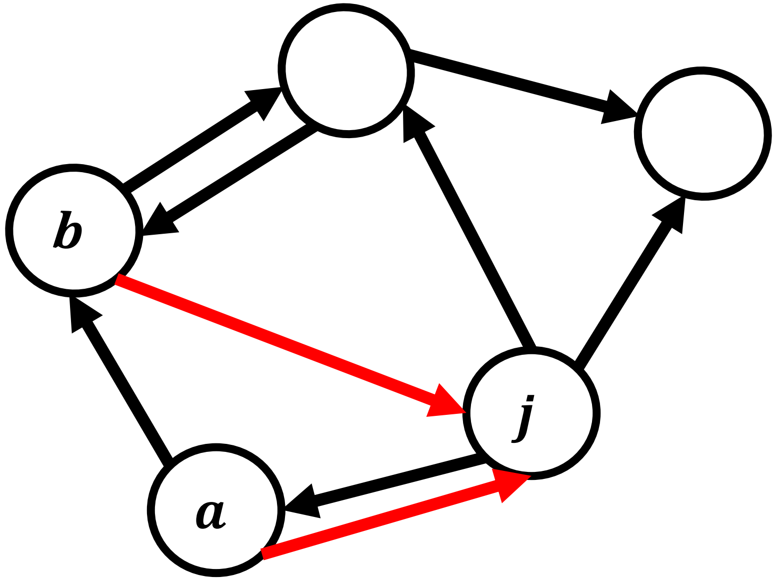
Directed graph $G = (V, E)$

- V: player set $\longrightarrow$ $|V| = n$
- E: captures interactions

Player $\boldsymbol{j} \in [\boldsymbol{n}]$ participates in game with her in-neighbours

[a] Graphical models for game theory. Kearns, Littman, Singh. 2001

Graphical games [a]



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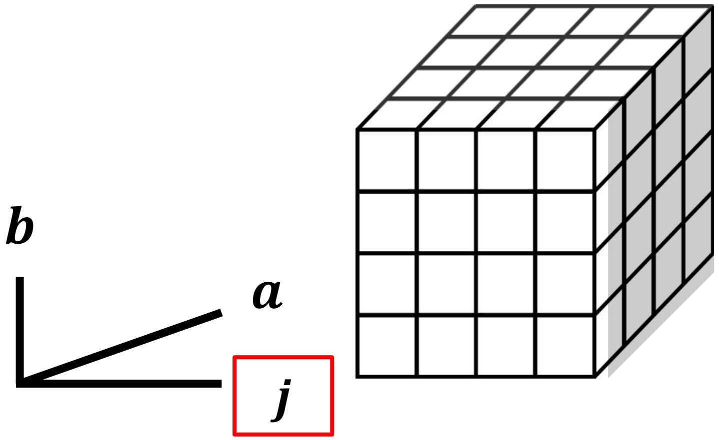
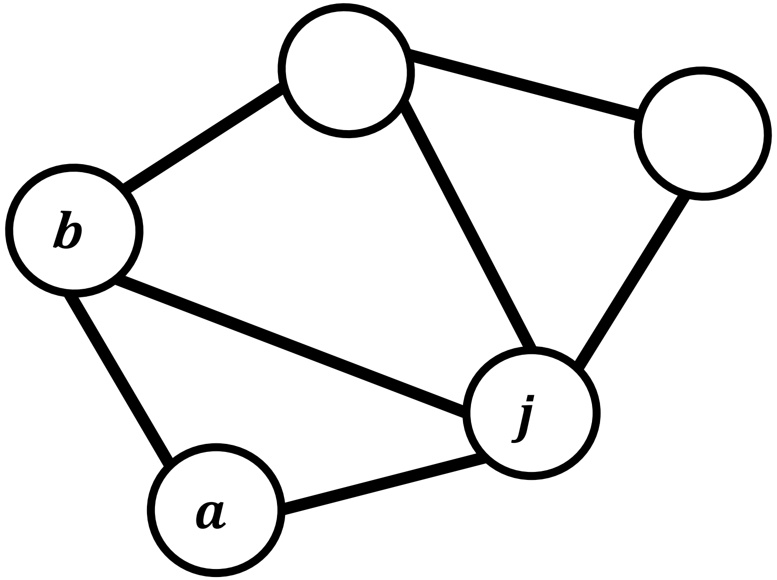


Input (m-action, d-in-degree): $\boldsymbol{n \cdot m^{d+1}}$ payoff entries

(succinct representation)

[a] Graphical models for game theory. Kearns, Littman, Singh. 2001

Class of graphical games: polymatrix [a]



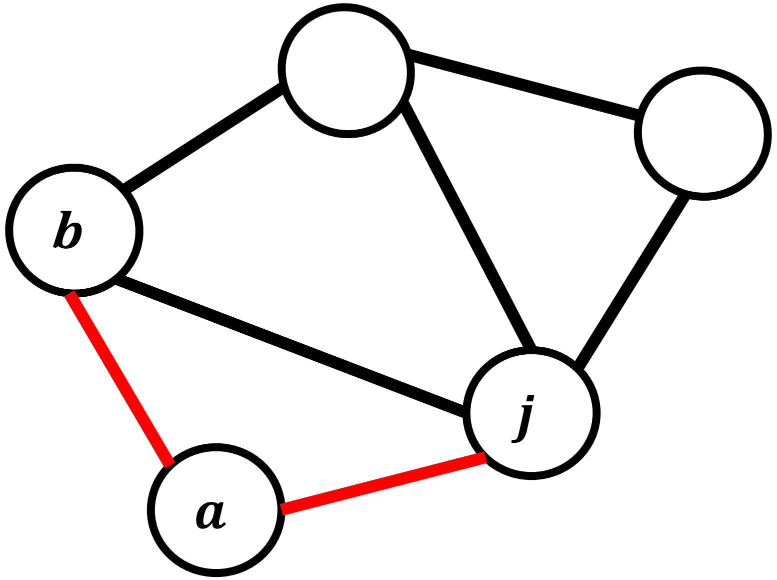
Undirected graph $G = (V, E)$

- V: player set $\longrightarrow$ $|V| = n$
- E: captures **pairwise** interactions

Player $\boldsymbol{j} \in [\boldsymbol{n}]$ participates in bimatrix games, one with each of her neighbours

[a] Equilibrium points in polymatrix games. Janovskaja. 1968

Class of graphical games: polymatrix [a]

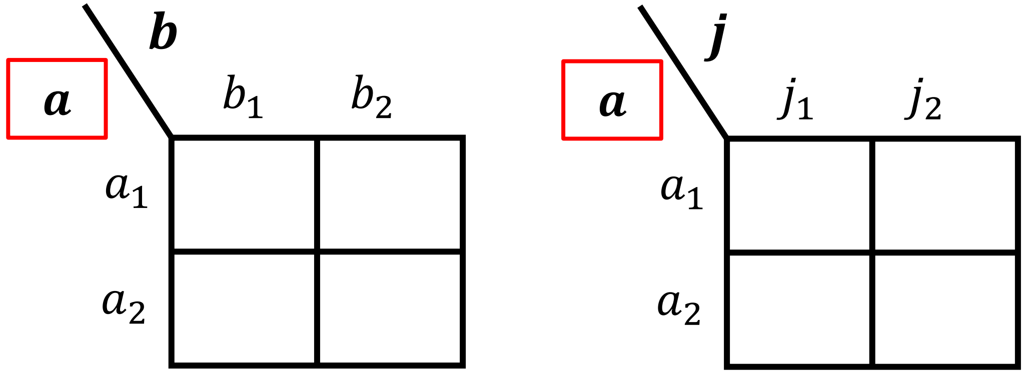


Undirected graph $G = (V, E)$

- V: player set $\longrightarrow$ $|V| = n$
- E: captures **pairwise** interactions

Player $j \in [n]$ participates in bimatrix games, one with each of her neighbours

- ***j*'s payoff: sum of bimatrix games payoffs**



Input (m-action, d-degree): $\boldsymbol{n \cdot d \cdot m^2}$ payoff entries

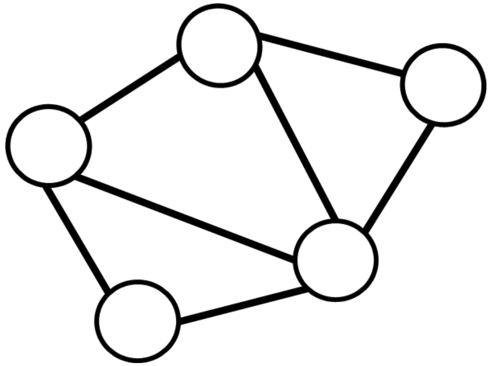
(succinct representation)

[a] Equilibrium points in polymatrix games. Janovskaja. 1968

Polymatrix games:
Algorithms and Complexity

Hardness results for polymatrix

- **Polymatrix games:**



➢ **PPAD-complete** [a], [b], [c]

END-OF-LINE $\leq_p$ DISCRETE-BROUWER $\leq_p$ GENERALIZED-CIRCUIT $\leq_p$ $\leq_p$ $\boldsymbol{\varepsilon}$-**WSNE-POLYMATRIX** $\leq_p$ $\boldsymbol{\varepsilon}$–**NE-POLYMATRIX** $\leq_p$ 2-NASH

- **In fact**:
  - ✓ $\boldsymbol{\varepsilon} = 1/\exp(\mathrm{N})$ [a]
  - ✓ $\boldsymbol{\varepsilon} = 1/\mathrm{poly}(\mathrm{N})$ [b]
  - ✓ $\boldsymbol{\varepsilon} = \mathrm{const}$ (of the order 10^{-8}) [c]
    - even for 2-action, degree-3, bipartite graphs

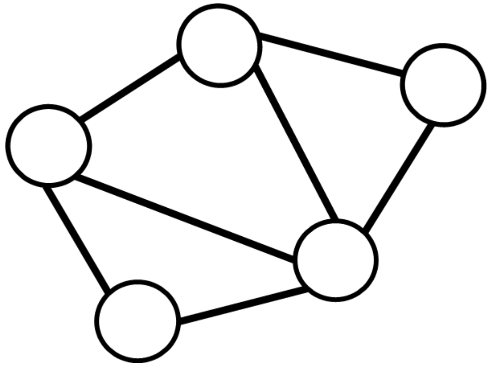
[a] The complexity of computing a Nash equilibrium. Daskalakis, Goldberg, Papadimitriou. 2008

[b] Settling the complexity of computing two-player Nash equilibria. Chen, Deng, Teng. 2009

[c] Inapproximability of Nash Equilibrium. Rubinstein. 2014

Hardness results for polymatrix

- **Polymatrix games:**

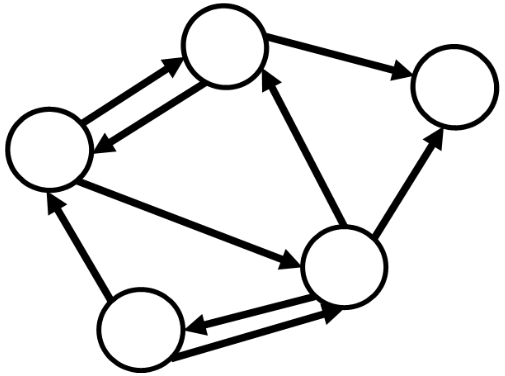


➤ **PPAD-complete** [a], [b], [c]

END-OF-LINE $\leq_p$ DISCRETE-BROUWER $\leq_p$ GENERALIZED-CIRCUIT $\leq_p$
$\leq_p$ **ε-WSNE-POLYMATRIX** $\leq_p$ **ε–NE-POLYMATRIX** $\leq_p$ 2-NASH

- **In fact**:
  - ✓ $\boldsymbol{\varepsilon} = 1/\exp(\mathrm{N})$ [a]
  - ✓ $\boldsymbol{\varepsilon} = 1/\mathrm{poly}(\mathrm{N})$ [b]
  - ✓ $\boldsymbol{\varepsilon} = \mathrm{const}$ (of the order 10^{-8}) [c]
    - even for 2-action, degree-3, bipartite graphs

- **Graphical games:**



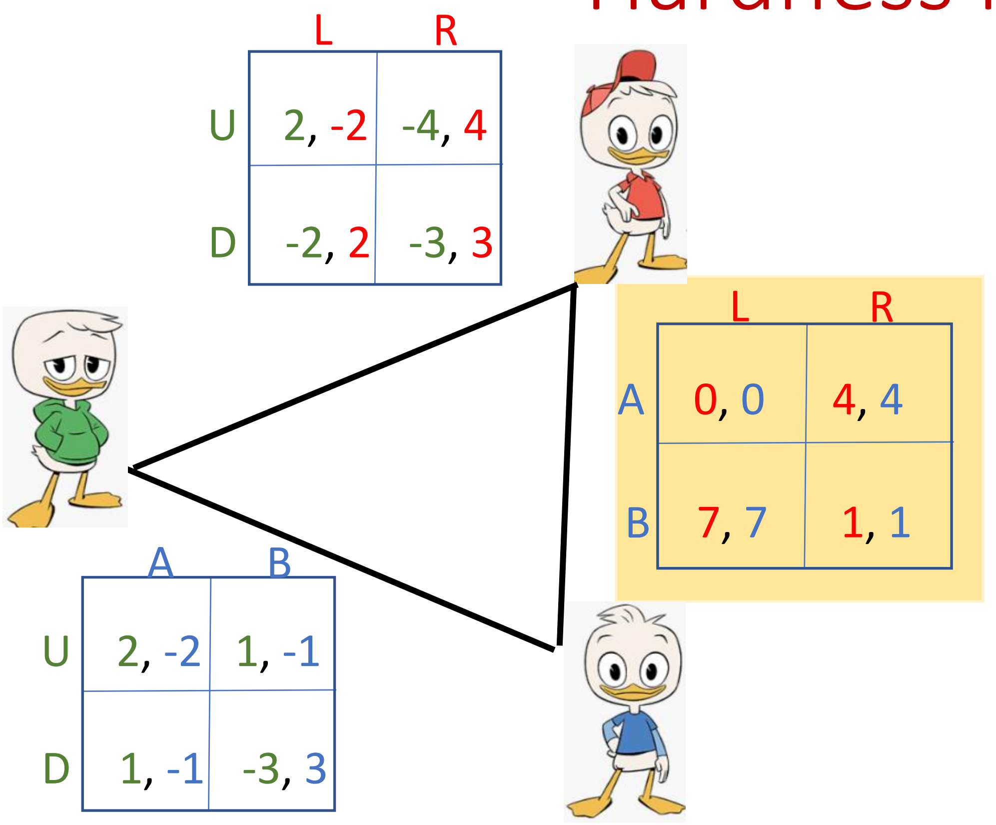
➤ **PPAD-complete:** poly-time reduction since actions and degree are constant

[a] The complexity of computing a Nash equilibrium. Daskalakis, Goldberg, Papadimitriou. 2008

[b] Settling the complexity of computing two-player Nash equilibria. Chen, Deng, Teng. 2009

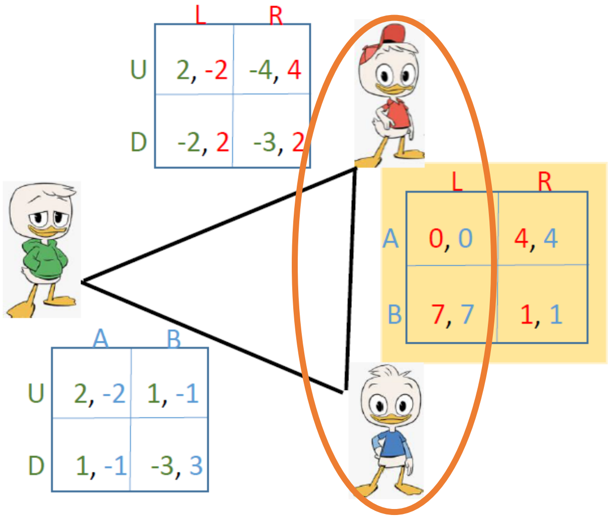
[c] Inapproximability of Nash Equilibrium. Rubinstein. 2014

Hardness results for polymatrix



Group-wise zero-sum polymatrix games

- ➢ Partition the players into groups
- ➢ Players in the same group play a coordination game
- ➢ Players in different groups play zero-sum games

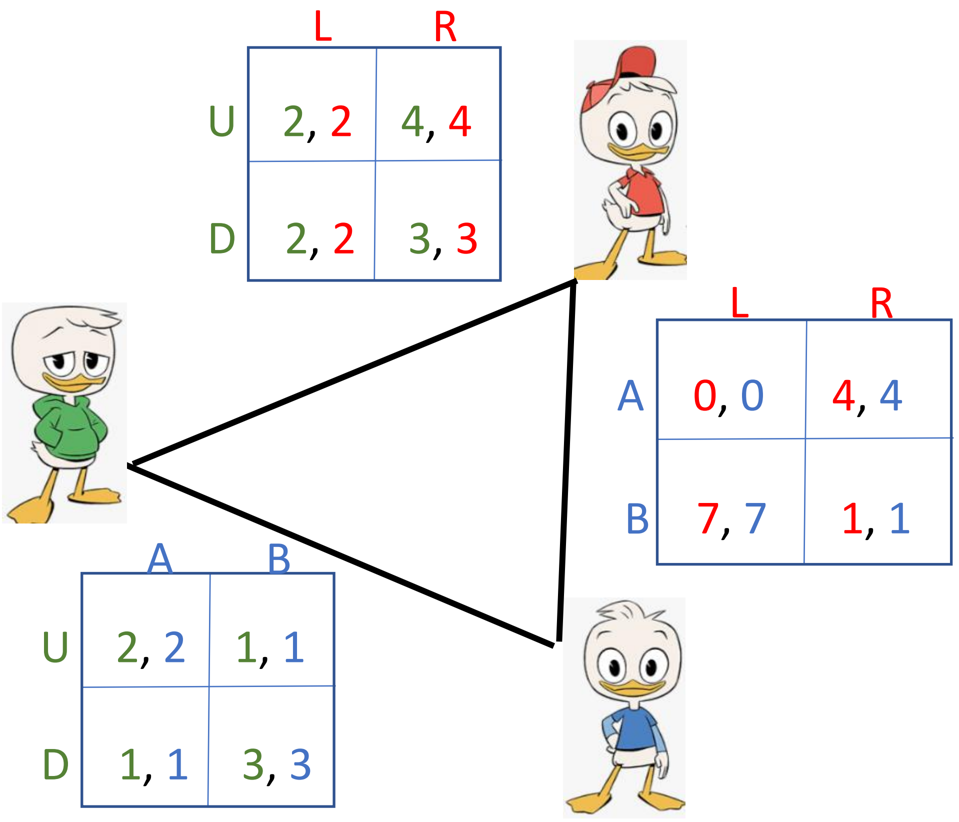


- Finding an NE in polymatrix games with zero-sum and coordination games on the edges is PPAD-complete [a]
- Group-wise zero-sum games are PPAD-complete even with three groups of players [a]

❖ **Open problem**: Group-wise zero-sum with two groups?

[a] On Minmax Theorems for Multiplayer Games. Cai, Daskalakis. 2011

Hardness results for polymatrix



Coordination-only polymatrix games [a]

- Find a **pure Nash equilibrim: PLS-complete**
- Find a **mixed Nash equilibrim: in PLS ∩ PPAD**

- **Open problem**: Complexity of (mixed) NE

[a] On Minmax Theorems for Multiplayer Games. Cai, Daskalakis. 2011

Hardness results: constrained NE in polymatrix

It is NP-complete to decide whether there is a strategy profile with sum of payoffs $\boldsymbol{u}$ even in polymatrix games with:

- degree 3, bipartite, planar graph
- at most 3 actions per player [a]

For any $\varepsilon \in [0,1]$, it is NP-complete to decide whether a polymatrix game has an ε-NE with sum of payoffs $\boldsymbol{u}$ [a]

For any $\varepsilon \in (0,1)$, it is NP-complete to decide whether a polymatrix game possesses a constrained ε-WSNE. This holds even for polymatrix games with:

- degree 3, bipartite, planar graph
- at most 7 actions per player [a]

[a] Computing Constrained Approximate Equilibria in Polymatrix Games. Deligkas, Fearnley, Savani. 2017

Algorithms for polymatrix

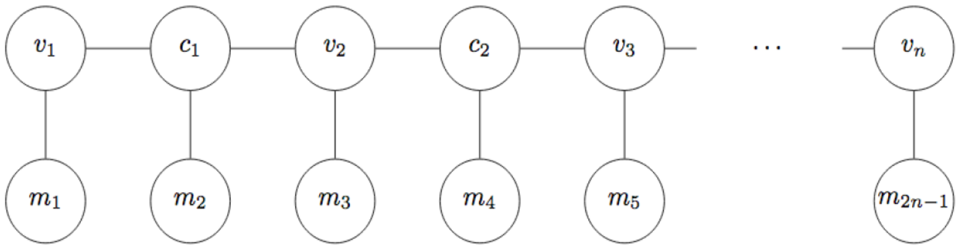
$\left(\frac{1}{2} + \delta\right)$-NE of a polymatrix game can be found in time $poly\left(N, \frac{1}{\delta}\right)$, for any $\delta > 0$ [a]

Graph classes:

- Paths with two actions: polynomial time [e]
- Cycles with two actions: polynomial time [e]
- Trees with constant actions
  - QPTAS [f]
  - FPTAS [g]
- Bounded treewidth: QPTAS [d]

All QPTASs results use the same underlying principle as [b], [c]

- Constant pathwidth: PPAD-hard [e]
- Sparse, win-lose, 2 actions: PPAD-hard [h]
- Trees, 20 actions, **exact** NE: PPAD-hard [i]



[a] Computing Approximate Nash Equilibria of Polymatrix Games. Deligkas, Fearnley, Savani, Spirakis. 2014

[b] Playing large games using simple strategies. Lipton, Markakis, Mehta. 2003

[c] Approximating the existential theory of the reals. Deligkas, Fearnley, Melissourgos, Spirakis. 2018

[d] Computing Constrained Approximate Equilibria in Polymatrix Games. Deligkas, Fearnley, Savani. 2017

[e] Nash equilibria in graphical games on trees revisited. Elkind, Goldberg, Goldberg. 2006

[f] Approximating Nash equilibria in tree polymatrix games. Barman, Ligett, Piliouras. 2016

[g] Tractable algorithms for approximate Nash equilibria in generalized graphical games with tree structure. Ortiz, Irfan. 2017

[h] On the Approximation of Nash Equilibria in Sparse Win-Lose Multi-Player Games. Liu, Li, Deng. 2021

[i] Tree polymatrix games are PPAD-hard. Deligkas, Fearnley, Savani. 2020

Graphical/polymatrix games: Recent tight results

A new tool to show PPAD-hardness

The **Pure-Circuit** problem [a]**:**

Input: A Boolean circuit, with a twist:

- The circuit can have **cycles**
- Nodes take values in $\{\mathbf{0}, \mathbf{1}, \perp\}$, instead of just $\{0,1\}$
- In addition to the standard logical gates (NOT, OR, AND), the circuit can also have "**PURIFY**" gates

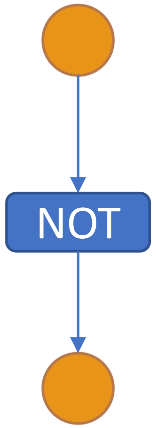
Goal: Assign a value (in $\{0,1,\perp\}$) to each node, such that all gates are "**satisfied**"

Pure-Circuit is PPAD-complete [a]

[a] Pure-Circuit: Strong Inapproximability for PPAD. Deligkas, Fearnley, Hollender, Melissourgos. 2022

Pure-Circuit: a new tool to show PPAD-hardness

NOT gate:



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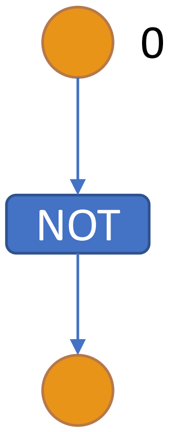

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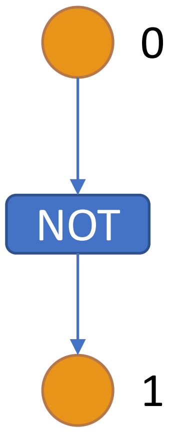

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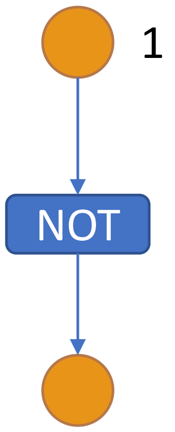

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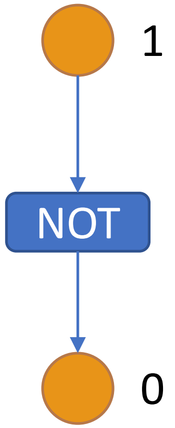

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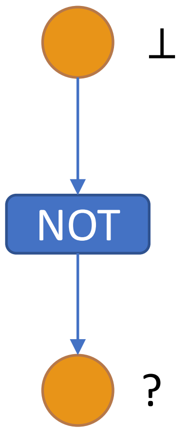

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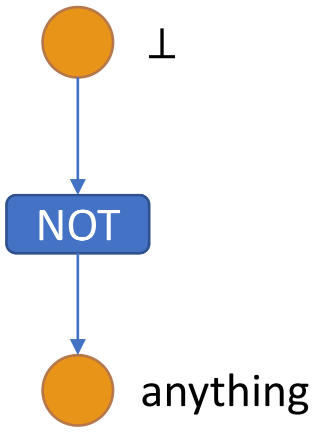

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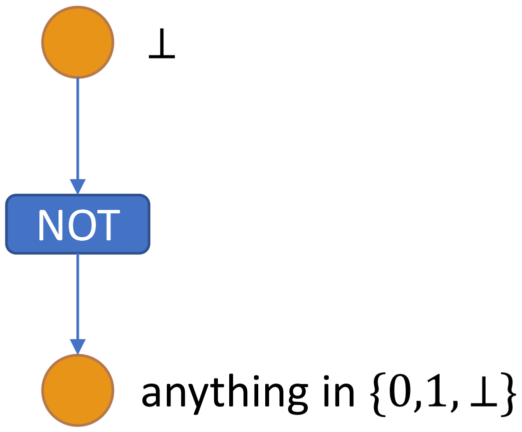
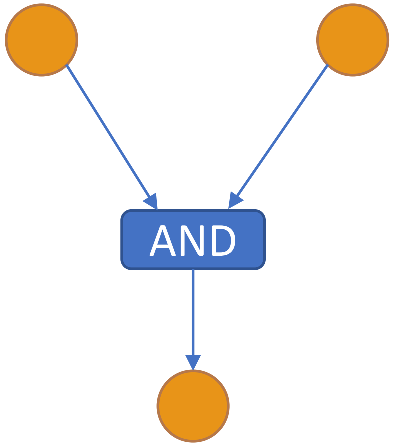


Pure-Circuit: a new tool to show PPAD-hardness

AND gate:



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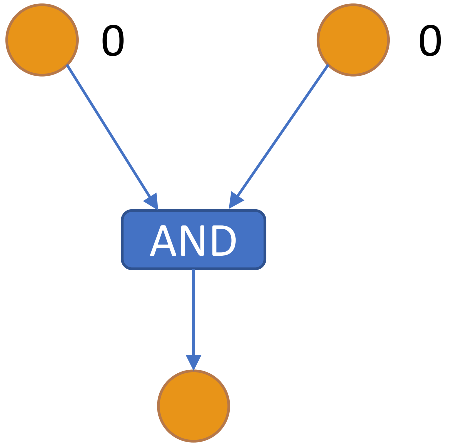

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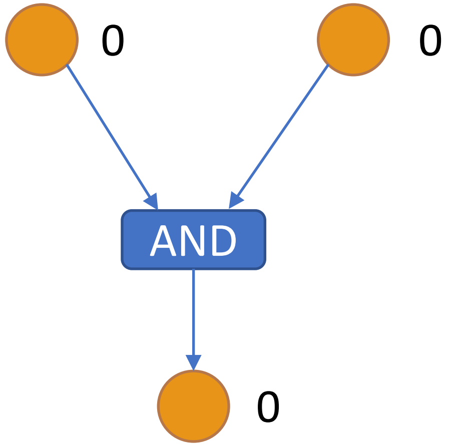

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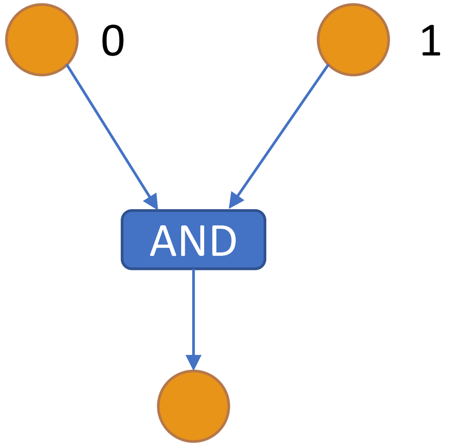

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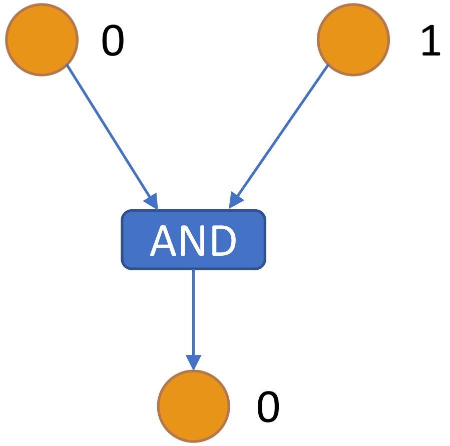

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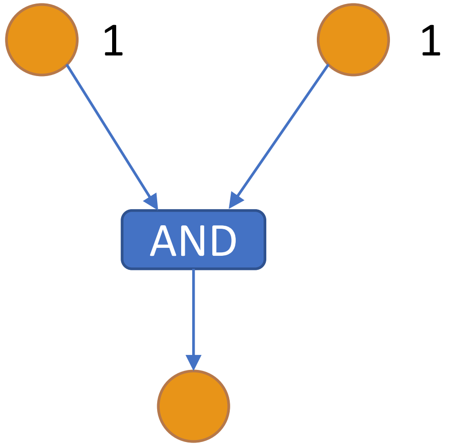

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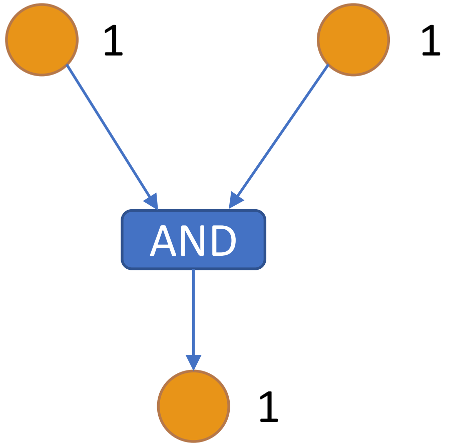

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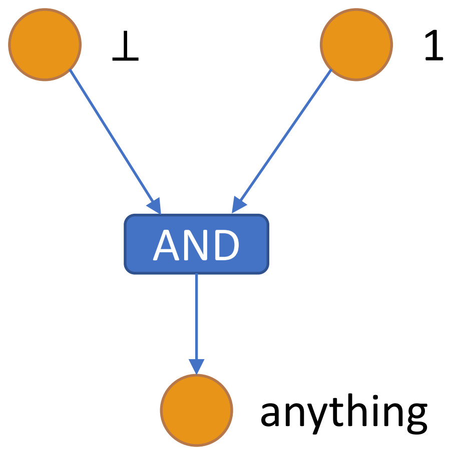

Pure-Circuit: a new tool to show PPAD-hardness

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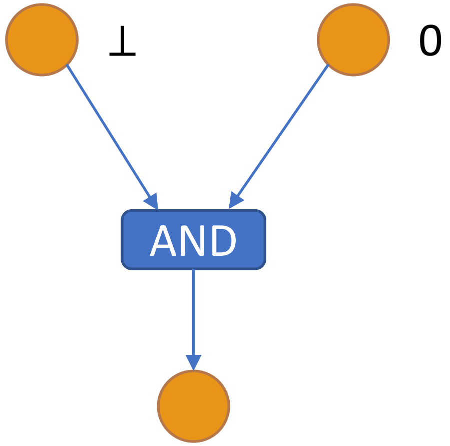

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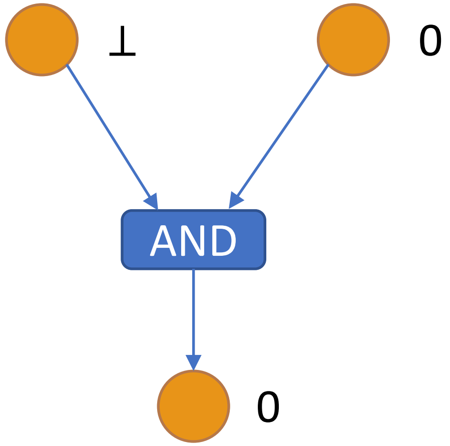

(with robustness!)

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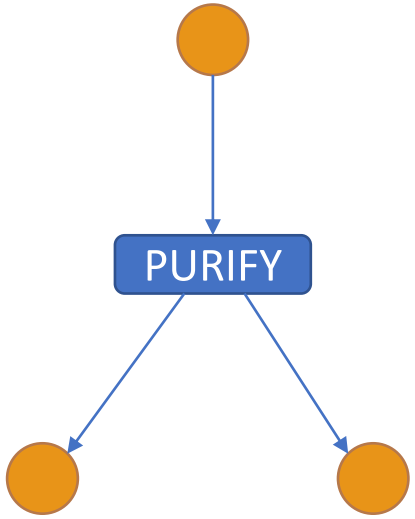

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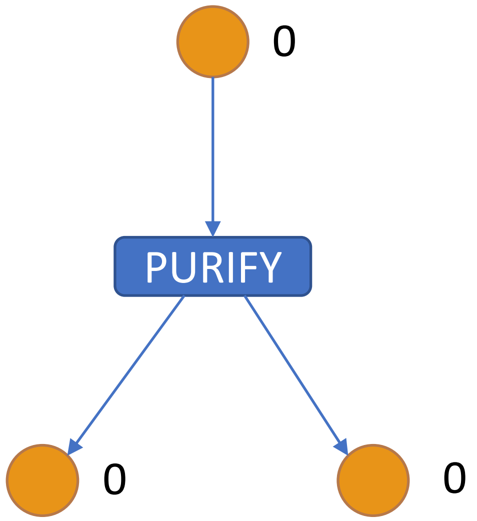

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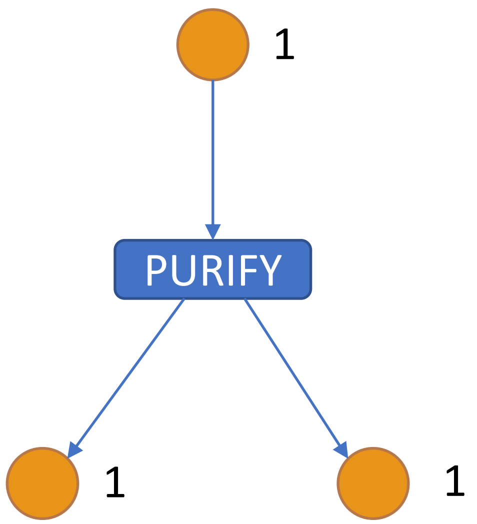

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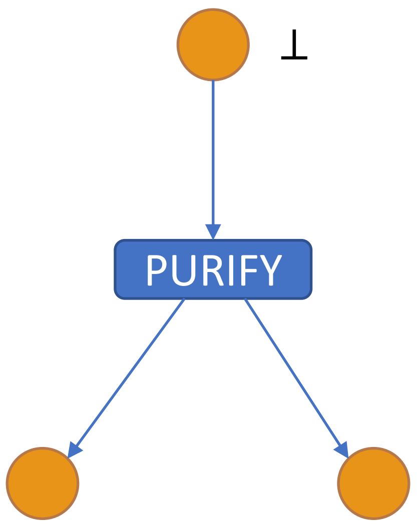

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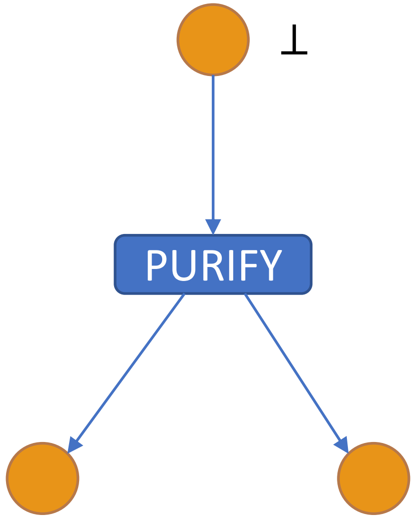

at least one in {0,1}

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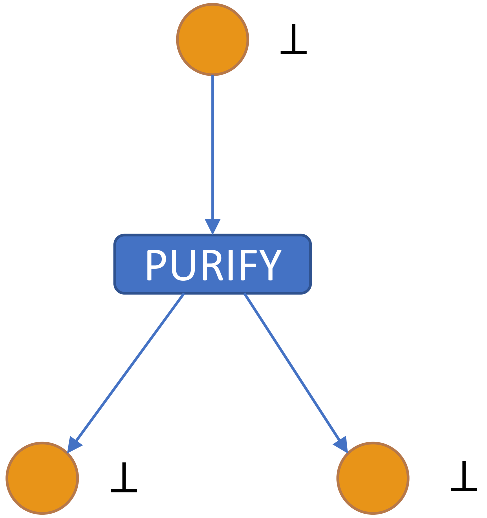

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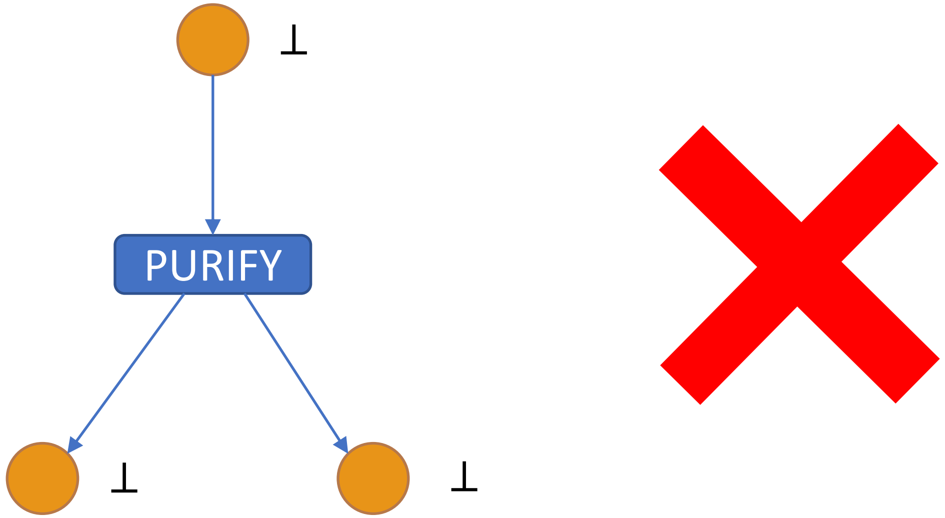

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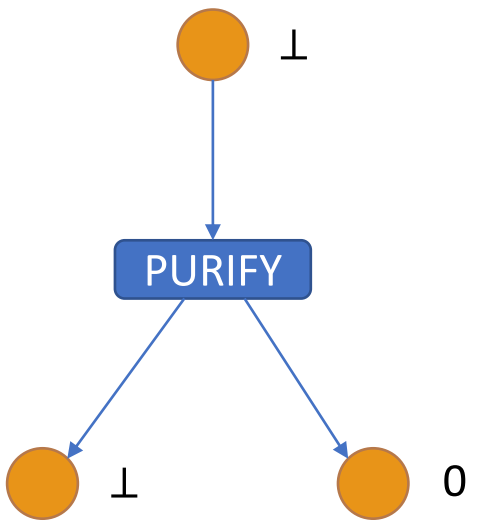


Pure-Circuit: a new tool to show PPAD-hardness

Pure-Circuit gates:

u	v
0	1
1	0
$\bot$	$\{0, 1, \bot\}$

NOT gate

u	v	w
1	1	1
0	$\{0, 1, \bot\}$	0
$\{0, 1, \bot\}$	0	0
Else		$\{0, 1, \bot\}$

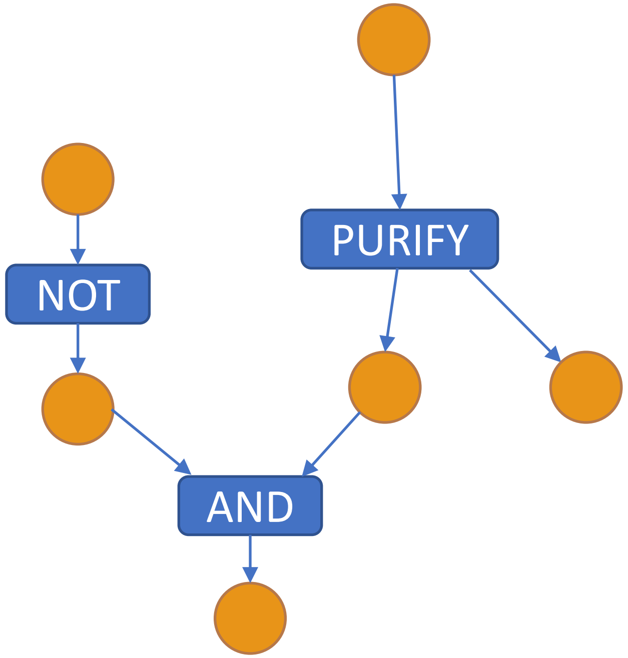
AND gate

u	v	w
0	0	0
1	1	1
$\bot$	At least one output in $\{0, 1\}$	

PURIFY gate

Pure-Circuit: a new tool to show PPAD-hardness

1. The circuit can have cycles



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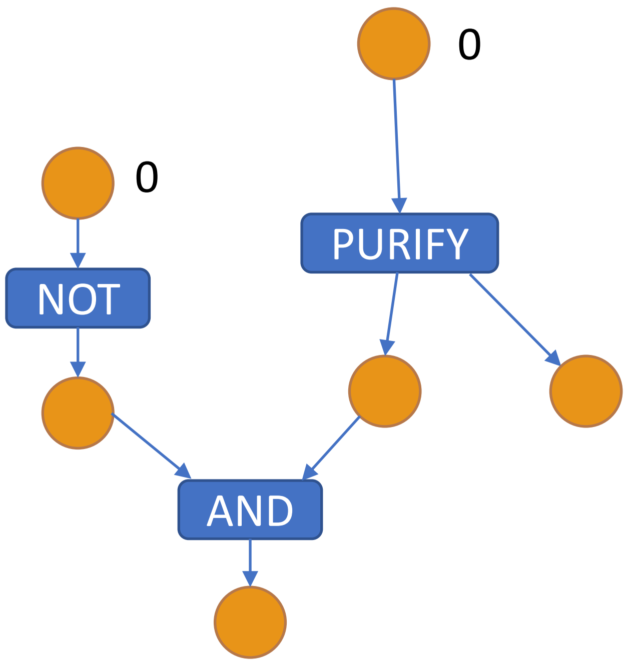

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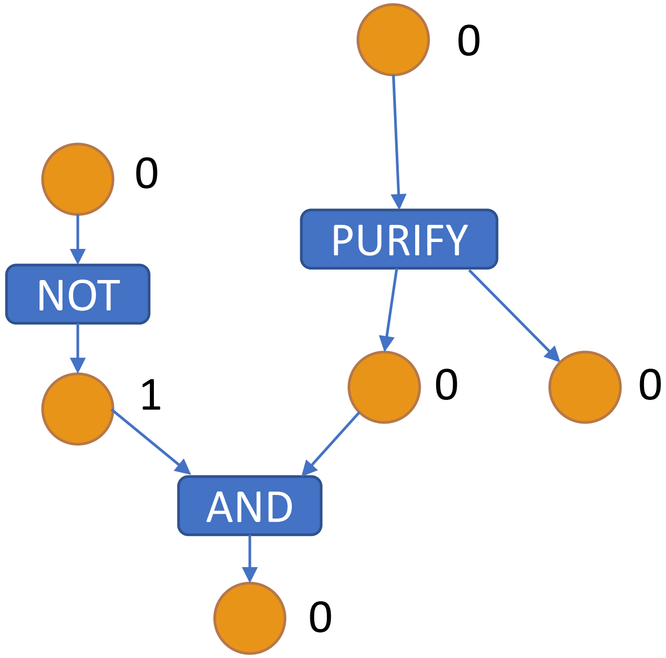

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2. Nodes take values in $\{0,1,\perp\}$, instead of just $\{0,1\}$

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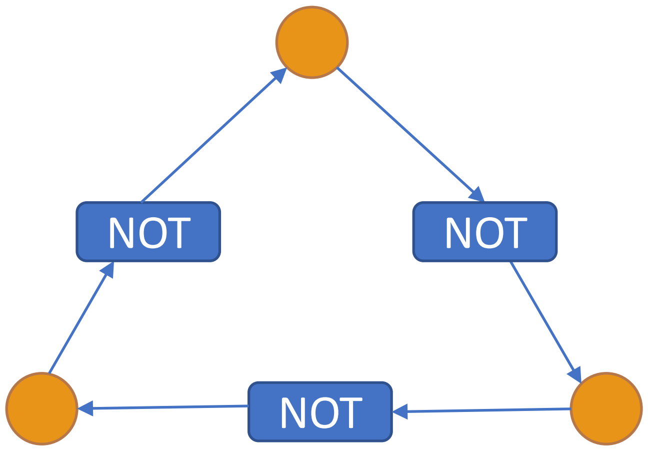

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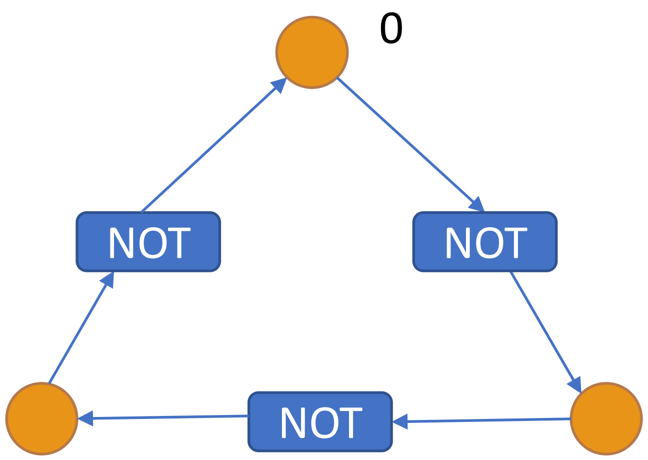

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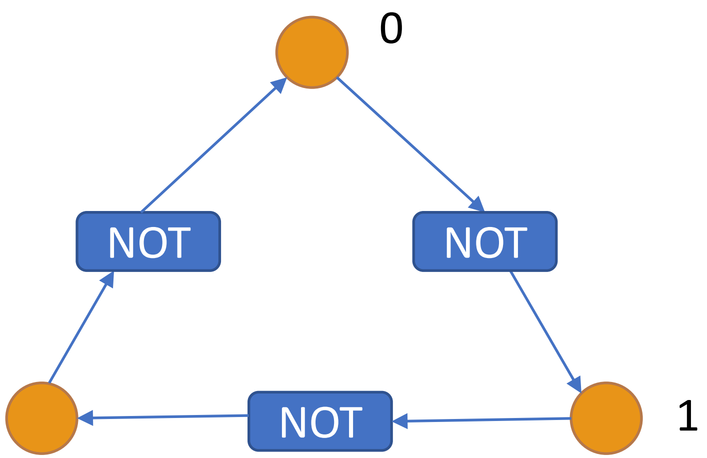

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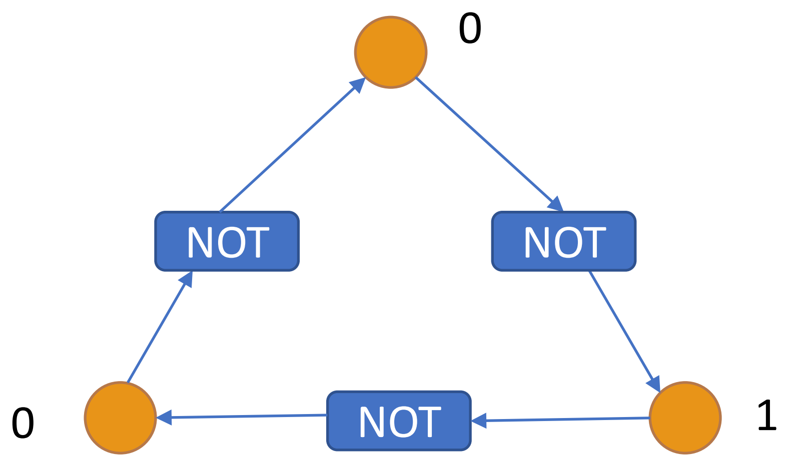

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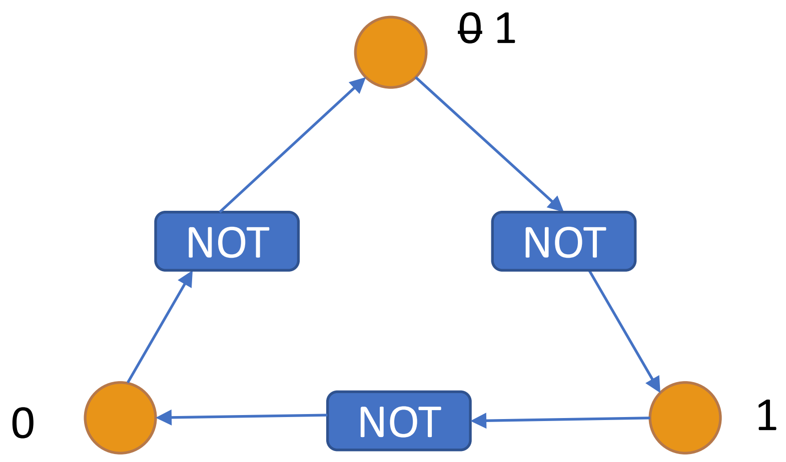

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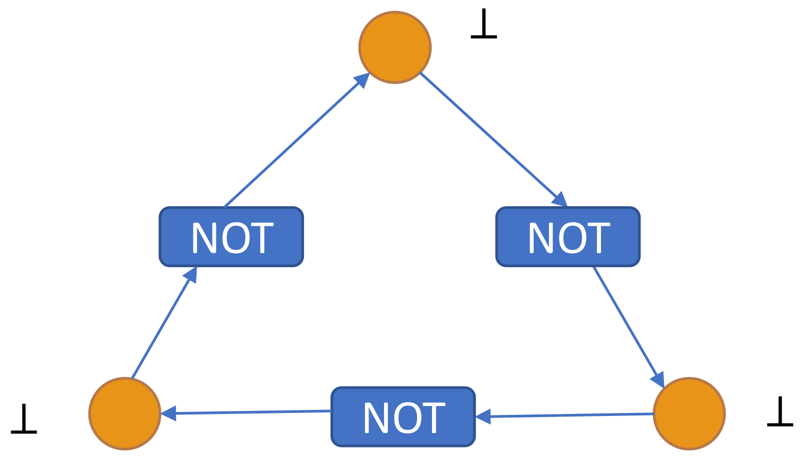

Pure-Circuit: a new tool to show PPAD-hardness

3. In addition to the standard logical gates (NOT, OR, AND), the circuit can also have “**PURIFY**” gates

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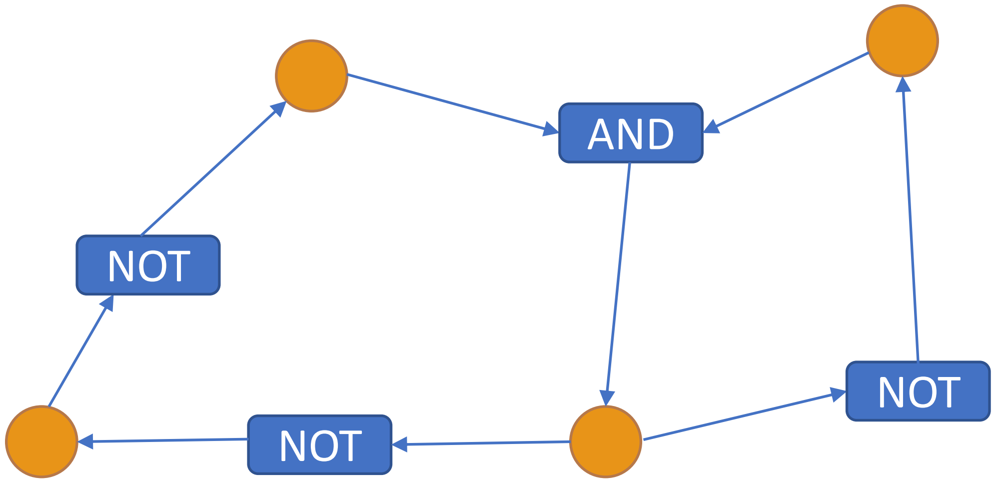

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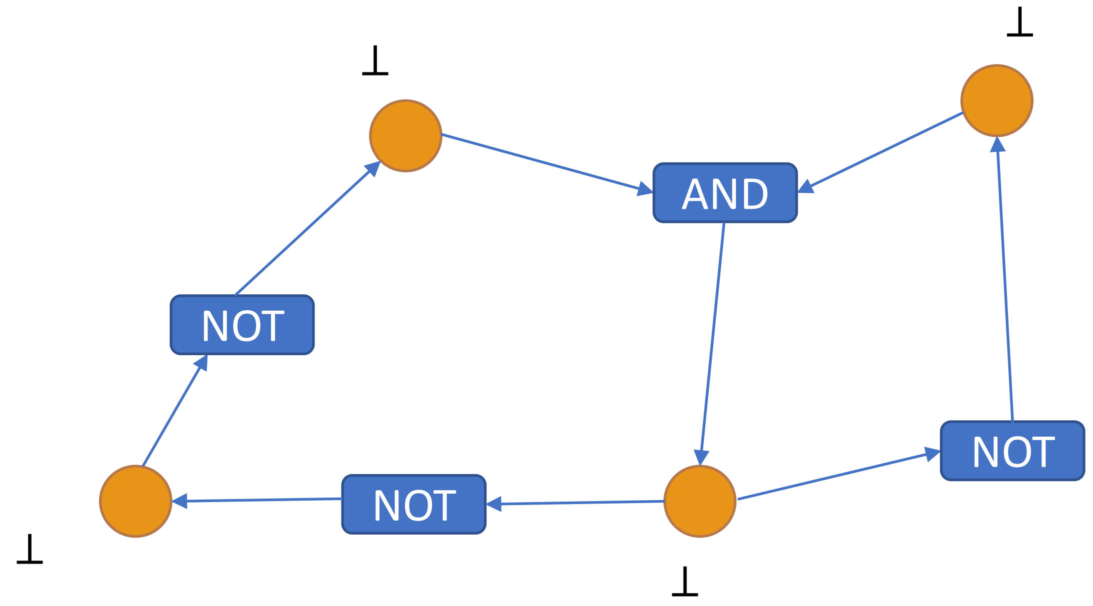

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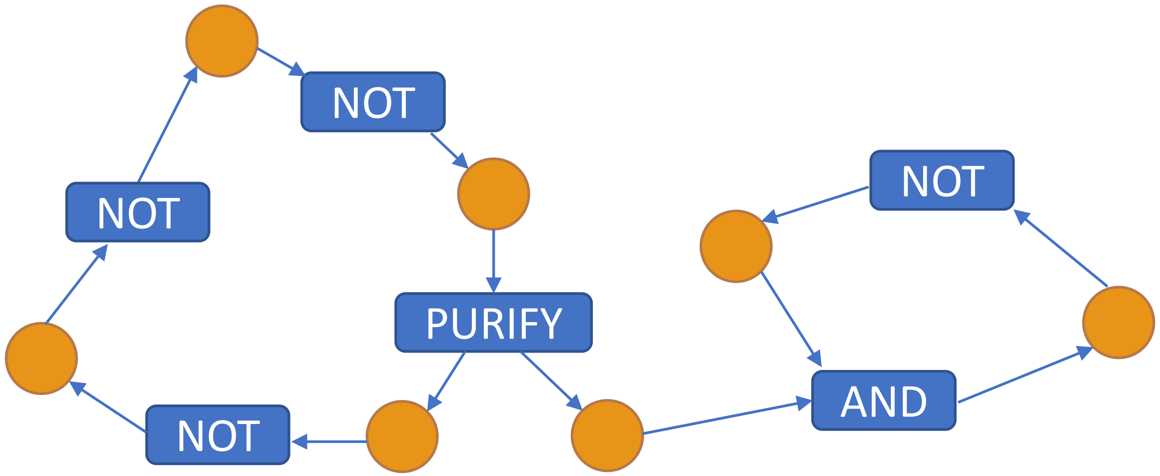

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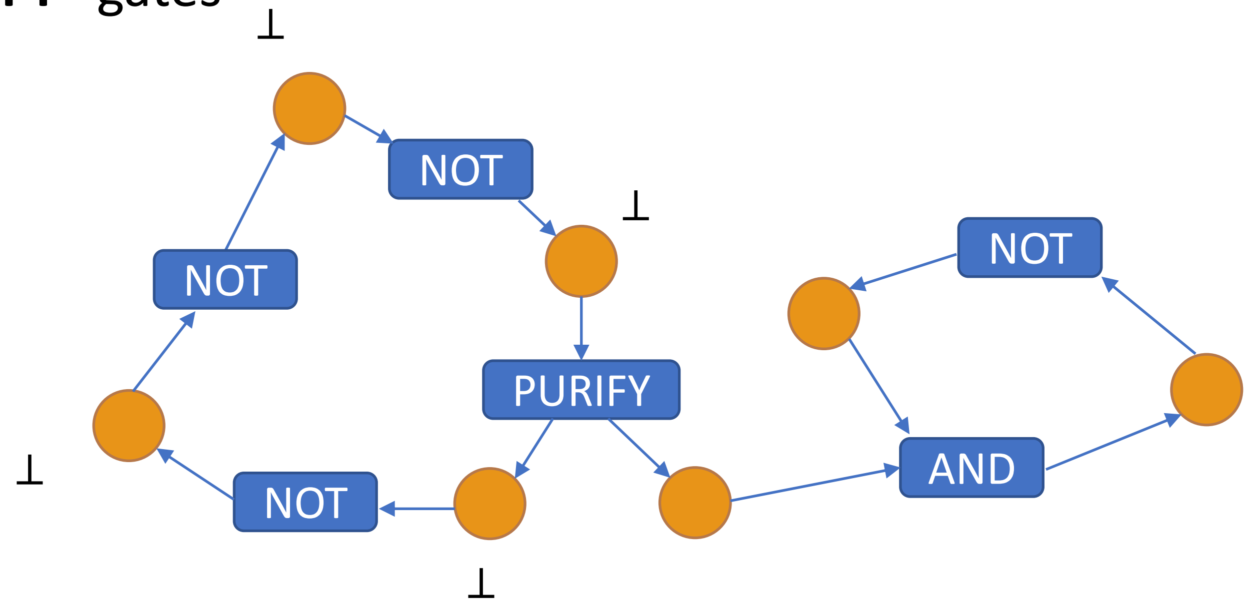

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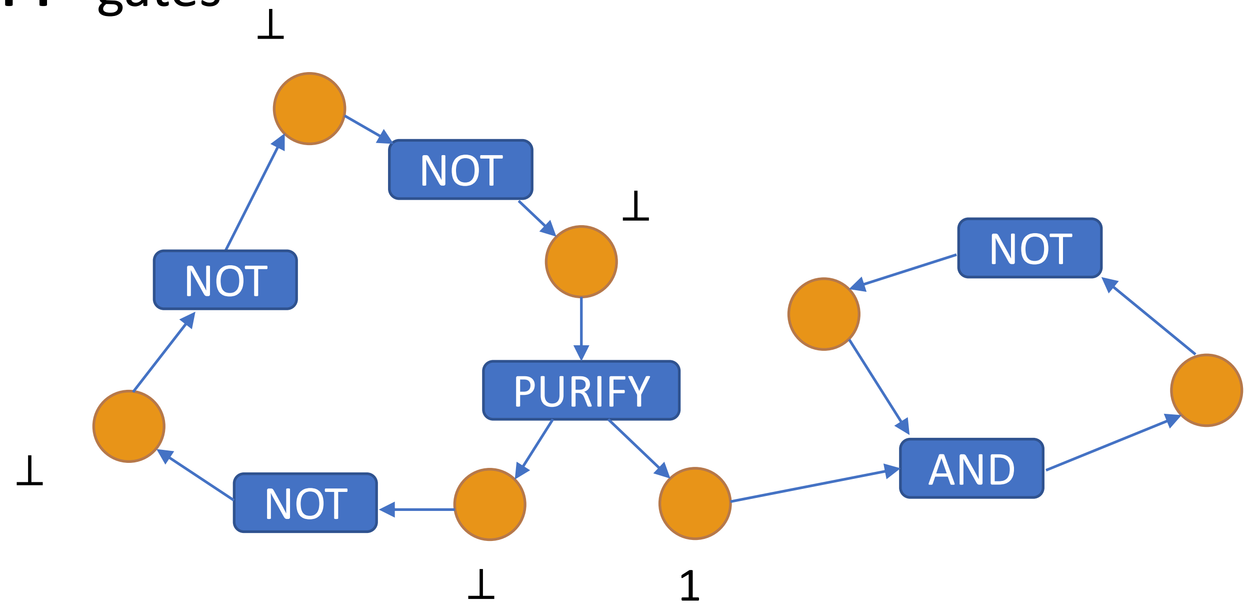

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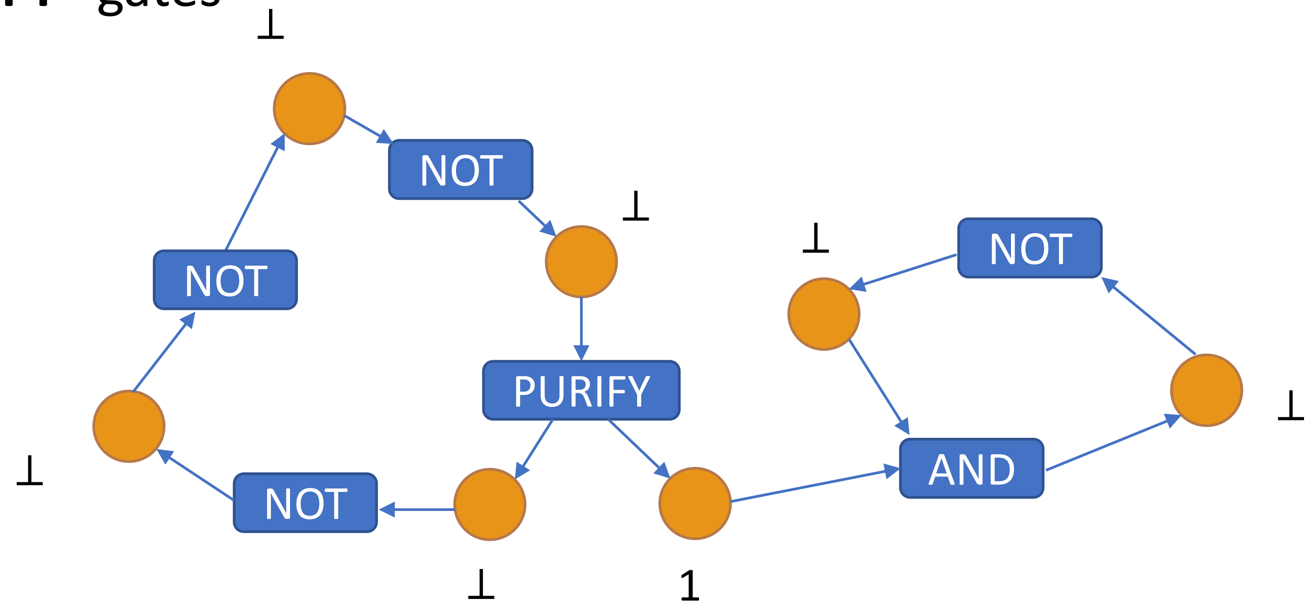

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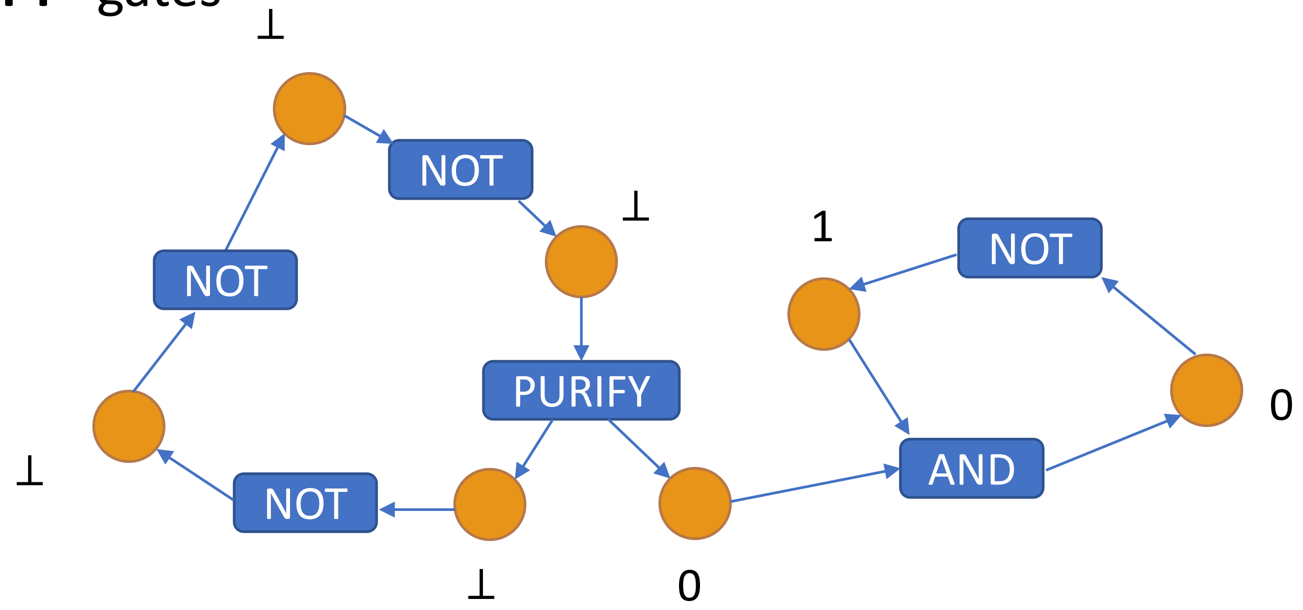
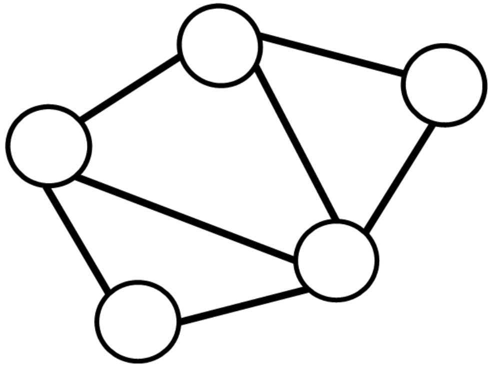


Stronger hardness results for ε-WSNE in polymatrix

- **Polymatrix games:**



➢ **PPAD-complete**

~~GENERALIZED-CIRCUIT~~ $\leq_p$ **ε-WSNE-POLYMATRIX**

PURE-CIRCUIT $\leq_p$ **ε-WSNE-POLYMATRIX**

- ✓ $\boldsymbol{\varepsilon} = 1/\exp(\mathrm{N})$ [a]
- ✓ $\boldsymbol{\varepsilon} = 1/\mathrm{poly}(\mathrm{N})$ [b]
- ✓ $\boldsymbol{\varepsilon} = \mathrm{const}$ (of the order 10^{-8}) [c]
  - even for 2-action polymatrix on bipartite graphs
- ✓ $\boldsymbol{\varepsilon} < \mathbf{1/3}$ [d]
  - **even for 2-action, degree-3, bipartite graphs**

Idea:
- replace each vertex of the Pure-Circuit graph by a player
- The player has 2 actions **zero, one** and his payoffs simulate the function of the corresponding gates
- **zero** $= \mathbf{0}$, **one** $= \mathbf{1}$, **mix** $= \perp$

[a] The complexity of computing a Nash equilibrium. Daskalakis, Goldberg, Papadimitriou. 2008
[b] Settling the complexity of computing two-player Nash equilibria. Chen, Deng, Teng. 2009
[c] Inapproximability of Nash Equilibrium. Rubinstein. 2014
[d] Pure-Circuit: Strong Inapproximability for PPAD. Deligkas, Fearnley, Hollender, Melissourgos. 2022

A simple algorithm for $1/3$-WSNE in 2-action polymatrix

$1/3$-WSNE algorithm for 2-action polymatrix games [a]**:**

1. Find a player such that one of its two actions is always a $1/3$-best-response (no matter what the other players play). Fix the player's strategy to that action, and remove the player from the game.
2. Repeat Step 1 until no such player exists anymore.
3. For the remaining players, have them mix uniformly between their two actions, i.e., have them play $\left(\frac{1}{2}, \frac{1}{2}\right)$.

→ Can show that this always yields a $1/3$-WSNE (by a simple direct computation)

PPAD/P dichotomy at $\varepsilon = 1/3$

- **Open problem**: At what ε is there a dichotomy for 3-action polymatrix ε-WSNE? What about more actions?

[a] Pure-Circuit: Strong Inapproximability for PPAD. Deligkas, Fearnley, Hollender, Melissourgos. 2022

Stronger hardness results for ε-NE in polymatrix

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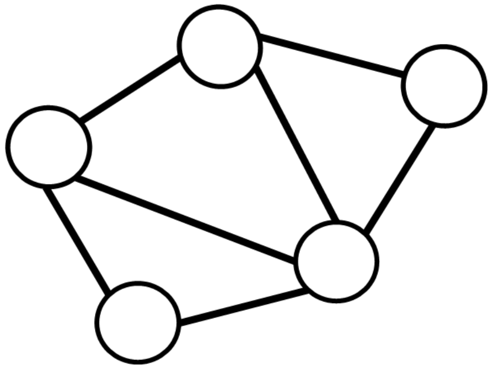

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  - even for 2-action polymatrix on bipartite graphs
- ✓ $\boldsymbol{\varepsilon} < \mathbf{0.088}$ [d]
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$1/5$-NE algorithm for 2-action polymatrix games:

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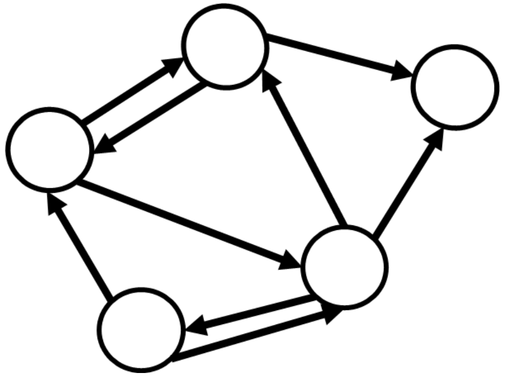
→ Can show that this always yields a **$1/5$-NE** (by a simple direct computation)

There is a gap in ε: $0.088 - 0.2$

❖ **Open problem**: Can we close this gap?

Tight hardness results for ε-WSNE / ε-NE in graphical

- **Graphical games:**



➢ **PPAD-complete**

~~GENERALIZED-CIRCUIT~~ $\leq_p$ **ε-WSNE-GRAPHICAL**
~~GENERALIZED-CIRCUIT~~ $\leq_p$ **ε-NE-GRAPHICAL**

PURE-CIRCUIT $\leq_p$ **ε-WSNE-GRAPHICAL**
PURE-CIRCUIT $\leq_p$ **ε-NE-GRAPHICAL**

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  - even for 2-action polymatrix on bipartite graphs
- ✓ $\boldsymbol{\varepsilon} < \mathbf{1}$ **for ε-WSNE** [d] - Any profile is 1-WSNE
- ✓ $\boldsymbol{\varepsilon} < \mathbf{1/2}$ **for ε-NE** [d] - 2-action games: every player $\left(\frac{1}{2},\frac{1}{2}\right)$ is a 1/2-NE

PPAD/P dichotomy at
- $\varepsilon = 1$ for WSNE
- $\varepsilon = 1/2$ for NE

Idea:
- Similar idea to polymatrix – different enco…

❖ **Open problem**: At what ε is there a dichotomy for 3-action graphical ε-NE? What about more actions?

[c] Inapproximability
[d] Tight Inapproximability for Graphical Games. Deligkas, Fearnley, Hollender, Melissourgos. 2022

Discussion

- **Close the gaps** of approximability-inapproximability
- Find **new** meaningful classes of instances that admit "**efficient**" algorithms

Thank you!